An Outline of the Logic of my Proof of the Riemann Hypothesis

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K. Eswaran
Sreenidhi Institute of Science & Technology

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Abstract

In this note I will try to describe in the simplest possible terms the method I adopted to prove the Riemann Hypothesis. My purpose is to help the reader not only to understand the proof but also enable him/her to verify the work (if necessary) for his/her own satisfaction.

As far as clearly spelling out the outline of the logic of my proof of RH, this document is self-contained. This is more detailed than the earlier "A Pathway to the Riemann Hypothesis" uploaded on March 2019, and is a corrected and amended version of "Method adopted on Proof of RH" (uploaded in my Project-log of Researchgate on August 6, 2021).

Introduction

In the following pages I give the basic steps of my proof of the Riemann Hypothesis. This brief note describes the scheme and the essential steps in the proof of the RH I gave in my Main Paper [1] and which is reproduced in Expert Committee Report (ECR) [2]. There are several theorems involved for which I have given alternate proofs, but here I will only outline those which are simple yet rigorous. I realize that it is difficult to describe my method with all its details to a complete layperson. But what follows will enable those with some background in mathematics to understand and appreciate at least the gist of the proof. The notes given in Refs. [3] and [4] will, I think, clarify many points.

1 Step 1:

First, I look at an analytic function, $F(s)$, whose poles exactly correspond to the non-trivial zeros of the zeta function, $\zeta(s)$, and use the techniques of complex function theory to analyze it. The function $F(s)$, defined as

$$F(s) = \frac{\zeta(2s)}{\zeta(s)}, \tag{1}$$

has these following properties: $F(s)$ has poles at exactly the same positions as $\zeta(s)$ has its non-trivial zeros in the critical region. Furthermore, the trivial zeros, $s = -2n$ where n is a positive integer, cancel out and do not appear as poles in $F(s)$. Also, $F(s)$ exhibits one additional pole which corresponds to the simple pole in $\zeta(2s)$, but this does not affect the analysis because it occurs on the critical line $s = 1/2$.

$F(s)$ is analytic in the region $Re(s) > 1$ and is there given by the infinite series:

$$F(s) = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}, \tag{2}$$

where the Liouville function $\lambda(n) = (-1)^{\Omega(n)}$, where $\Omega(n)$ is the number of prime factors of n, multiplicities included, with $\lambda(1) \equiv 1$.

2 Step 2:

In this step the necessary and sufficient conditions for the analyticity of $F(s)$ in the region $Re(s) < 1$ is determined. A technique previously used by Littlewood, for similarly determining the behaviour of the function $1/\zeta(s)$ by analytic continuation, is used here to examine the behaviour of $F(s)$. The summatory function $L(N)$ which is defined by :

$$L(N) = \sum_{n=1}^{N} \lambda(n) \tag{3}$$

plays a crucial role in determining the position of the poles of $F(s)$, and thereby the zeros of $\zeta(s)$, in the critical region $0 < Re(s) < 1$. Littlewood's theorem, when applied to $F(s)$, states that the asymptotic behaviour of $L(N)$ for large N determines the analyticity of $F(s)$, and that iff the behaviour is such that

$$|L(N)| \equiv |\sum_{n=1}^{N} \lambda(n)| < C\, N^{a+\epsilon} \qquad (for\, large\, N) \tag{4}$$

where $1/2 \leq a < 1$, $\epsilon > 0$, and C is a constant, $F(s)$ will be analytic in the region $a < Re(s)$. This is the crucial result as far as RH is concerned because if one can determine that actually $a = 1/2$ in (4) then the Riemann Hypothesis is proved (see below). Inequality (4) is proved to be true in Section 5.1 of my Main Paper. (See Equation (5.24), but replace $G(x)$ by $L(x)$.) An alternative way of writing Eq. (4) is that $F(s)$ will be analytic in the region $a < Re(s)$ iff

$$\underset{N\to\infty}{Lim} \,|\frac{L(N)}{N^{a+\epsilon}}| = 0, \qquad \epsilon > 0 \tag{4a}$$

We know, from Riemann's own results (and the Prime Number theorem), that the non-trivial zeros of $\zeta(s)$ all occur in the critical strip $0 < Re(s) < 1$, and will be symmetrically placed in complex conjugate pairs around the critical line $Re(s) = 1/2$. By examining (4), we can arrive at the following conclusion: If we can prove that $a = 1/2$, it means that it is possible to analytically continue $F(s)$ leftwards from $Re(s) = 1$ right up to the critical line $Re(s) = 1/2$ without meeting poles of $F(s)$. But, since a pole of $F(s)$ is a nontrivial zero of $\zeta(s)$, this means the zeros of the the zeta function cannot lie in $1/2 < Re(s) < 1$ and, by symmetry, in $0 < Re(s) < 1/2$, and so *must* lie on $Re(s) = 1/2$, thus proving RH.

So we conclude that to show that $a = 1/2$ in (4) or (4a) is to prove the RH. To do this we proceed as follows.

3 Step 3:

The necessity that (4) or (4a) must be satisfied, for the Riemann Hypothesis to be true, imposes very severe restrictions on the behaviour of the sequence of the Liouville functions: $\{\lambda(1), \lambda(2), \lambda(3), ...\}$.

As mentioned, $\lambda(n)$ is defined as: $\lambda(1) = 1$ and for $n > 1 : \lambda(n) = (-1)^{\Omega(n)}$ and is determined by factorizing n and finding $\Omega(n)$, the number of prime factors of n (multiplicities included). We already know that $\lambda(n)$ is fully determined by factorizing n and has a multiplicative property, namely: $\lambda(m \times n) = \lambda(m)\lambda(n)$, for all natural numbers m, n.

Now, for RH to be true, in (4) or (4a), the first N terms (where N is large) of the λ sequence must sum as:

$$| \lambda(1) + \lambda(2) + \lambda(3) + ... + \lambda(N) | < C \ N^{1/2+\epsilon} \tag{5}$$

The above equation brings to mind a similar relationship satisfied by another sequence of numbers $c(1), c(2), c(3), ...$, where $c(n)$ ($= \pm 1$ with equal probability and randomly) corresponds to the n^{th} step of a one-dimensional random walk in x. If $c(n) = +1$, it means the $n^{th}$is a unit step in the positive x-direction, and if $c(n) = -1$ it is in the negative x-direction. The $c(n)$'s can also be thought of as perfect coin tosses, if we replace, say, Heads by $+1$ and Tails by -1; so a N-step random walk can be thought of as a sum of a coin toss experiment where a coin is tossed N times. It is well known that for such a random-walk sequence, where the sum indicates the distance travelled from the starting position in N steps, that the magnitude of such a sum (assuming N is very large) satisfies the relationship:

$$| c(1) + c(2) + c(3) + ... + c(N) | < C' \ N^{1/2+\delta}, \tag{6}$$

where C' is a constant and δ is a small positive number which tends to zero[1] as $N \to \infty$. I shall refer to this as the square root law.

This is derived by using the standard Peano's Axioms (PA) Peano's Axioms and the assumptions that the random walk behaves in such a manner that:

[1] This result follows from the Law of the Iterated Logarithmof Khinchine and Kolmogorov, (see pp. 169-170 of ECR).

(a) Each step can be either in the positive direction or negative direction, i.e., in the n^{th} step $c(n)$ can be $+1$ or -1, with equal probability.

(b) The value of $c(n)$ is independent of all previous c's. In other words, knowing the previous consecutive M steps leading to the n^{th} step, yields no additional information on the value of the $(n+1)^{th}$. It follows that the Random walk is unpredictable, because by knowing any set of the previous values of the c's till $c(n)$, we cannot predict $c(n+1)$. Henceforth we will use *Unpredictability* to be synonymous with *Independence*. In particular, it follows that the sequence of steps cannot be periodic, that is, the pattern of steps cannot exhibit a repetitive, and thus predictable, pattern.

The two assumptions (a) and (b) along with Peano's Axioms are enough to derive (6). This has been shown by many researchers (e.g. See Chandrasekar (1943), Khinchine (1924) and Kolmogorov(1929), see references in Main Paper. Also the relevant part of S.Chandrasekhar's paper is reproduced in ECR pp. 59-64. Chapter 1 in pp. 60-61 contains his two assumptions). I will refer to the above conditions as (a) Equal Probability and (b) Independence.

It is most important to realize that only conditions (a) and (b) and PA are necessary to prove that the square root law in (6) is valid for large N. No other condition or assumption regarding the behaviour of a random walk (or coin tosses) is necessary.

3.1 The Argument:

As we have seen, Eq. (5) must be satisfied by the $\lambda(n)$ sequence *if the Riemann Hypothesis is TRUE*, which we deduce from Littlewood's Theorem. However, (5) needs be satisfied only for *large* N (this being the actual condition of Littlewood's theorem).

Turning to (6), note that there are many (actually, infinite, when $N \to \infty$) distinct sequences of random walks possible, as they can differ from each other at every step, or toss of the coin. For instance, if 1,000 random walkers, say, each take N steps, for sufficiently large N these 1,000 sequences will become uniquely different and can be thought of as 1,000 different *instances* of a random walk of N steps each. As they are also independent of each other nothing more can be said about their relationship with each other, except that on average the distance traveled in N steps will be around $C'\ N^{1/2}$ from the starting point.

If we wish to compare (6) with (5) there are several conceptual issues. First, the sequence in (5) is *deterministic* because $\lambda(n)$ can be precisely calculated for each value of n. However, we show that the $\lambda(n)$ sequence, nevertheless, satisfies conditions (a) and (b) mentioned above, and thereby behaves as a perfect Random Walk, in the limit of large N. Second, we have only one such $\lambda(n)$ sequence because there is only one natural number system. So we then consider this single sequence as *one instance* of a hypothetical random walk of N steps, when N is large, and investigate it in like manner.

Properties of the $\lambda-$sequence that need to be proved

(α) Given an arbitrarily large n chosen at random, there is an equal probability of $\lambda(n)$ being either $+1$ or -1.

(β) The value of $\lambda(n)$ is independent of all previous λ's for large n. Knowing the values of $\lambda(k)$ for any finite set of M consecutive values $k = n - M + 1$ to $k = n$, i.e. upto $\lambda(n)$ does not help us predict $\lambda(n + 1)$ (independence)[2].

It is to be noted that as, according to Littlewood's theorem, the RH is determined only by the asymptotic behaviour of the λ sequence, these need be shown in our case only for large N, i.e., $N \to \infty$. In particular, (β) can be interpreted to mean that $\lambda(n)$ should be independent of any finite number of preceding λ's.

Note that (α) and (β) are strict analogues of the (a) Equal Probability and (b) Independence conditions required for Random Walks, except that they need hold only for large N. ***If, by using the number theoretical properties of the integers (i.e., PA), it is somehow possible to prove that the λ sequence satisfies the rules of Equal probability and Independence for large N, then we will show that the λ-sequence is one particular instance of a Random Walk, as*** $N \to \infty$.

Given the above reasoning, if (α) and (β) are satisfied by the λ-sequence then *it is a fait accompli that all theorems of Random Walks must hold for the λ sequence, for $N \to \infty$. Otherwise there is a problem with the consistency of mathematics, pointing to an inconsistency in Peano's Axioms.* [Were this not so, one can argue thus: How is it that, in case of a random walk sequence $\{c(n)\}$, which satisfies (a) and (b), it can be proved by using PA that some result is true, but for the case of the λ-sequence $\{\lambda(n)\}$, which is shown by using PA to satisfy the identical conditions (α) and (β), the same result is *not* true?] This is emphasized here to underscore the important point that the deterministic nature of the $\lambda(n)$'s is not relevant to the statistical properties that it displays (see footnote 2). This reasoning, which is actually 'proof by contradiction', takes its final recourse, as always, to the consistency of mathematics to prove a theorem.

Hence the next step is to prove that the properties (α) and (β) hold for the $\{\lambda(n)\}$for large values of n (i.e., $n \to \infty$).

[2] The fact that the $\lambda(n)$ is actually deterministic plays no role in the proof of its independence, for large n. For example, given $\lambda(n)$ for some n, the formula $\lambda(m.n) = \lambda(m).\lambda(n)$,can determine the next predictable value $\lambda(2.n) = \lambda(2).\lambda(n) = -\lambda(n)$, but for large n, say $n = 10^{100}$, the integer $2n$ will be at a distance of 10^{100} from n making such a prediction statistically insignificant and irrelevant in the limit $n \to \infty$. Notice that the values of all the other $\lambda(k)$'s **in the range between** $k = 10^{100} + 1$ and $k = 2 \times 10^{100} - 1$ which are terms belonging to an extremely long sequence, are all independent of each other! This is because for any given k, if $\lambda(k)$ is known, the nearest value which can be deduced from this is $\lambda(2k) = -\lambda(k)$, but $2k$ is out of the above defined range and is increasingly far from k as $k \to \infty$.

4 Step 4: Proofs of Properties of the λ-sequence

In this step several theorems are proved using the number theoretical (arithmetical) properties of integers, primes and the unique factorization of integers to establish the properties (α) and (β) of the λ-sequence as listed above. These proofs are fairly straightforward and are done from first principles:

(α) On Equal Probabilities, is proved in theorem 3-B, Sec 5.2 of my Main Paper (p. 168 of Expert Commitee Report, (ECR)). For this purpose the concept of "Towers"[3] is used in the proof.,Sec 2, pp. 159-161 of ECR . An alternative proof by constuction of all prime products and induction is also given in a separate paper pp. 195-197 of ECR. A third proof, which follows from Littlewood's theorem but assumes the fact that there is no zero with $Re(s) = 1$ (proved in the Prime number Theorem) can also be derived (this is not given in the Main paper, but see pp. 13-14 of ECR). In fact, my second proof of Equal Probabilities is effectively a very short (two page) proof of the Prime Number Theorem! See footnote 6 on page 6 of Ref (4) in Reference List given below.

(β) Two separate proofs are given for unpredictability (independence). The first uses a purely arithmetical argument (see Sec. 11.2-11.4 page 173 of ECR). The second method uses the intuitive conclusion of Kurt Godel who said that every predictive function must be recursive. Using this, if we assume that one can predict $\lambda(n)$ given the (finite) M previous values of the λ's, it will inevitably result in the λ-sequence becoming cyclic, which in turn will imply (through Littlewood's theorem) that there are *no* zeros of $\zeta(s)$ in the critical strip, contrary to the known fact. In other words, no relationship such as $\lambda(n) = f(\lambda(n-1), \lambda(n-2), \lambda(n-3), ..., \lambda(n-M))$ is ever possible.[4] This makes the $\lambda(n)$ unpredictable and independent of finite numbers of previous values, as $n \to \infty$. This also demonstrates that the sequence $\{\lambda(n)\}$ is noncyclic (see Appendix III p. 172 of ECR).

Since it has now been proved that the sequence $\{\lambda(n)\}$ exhibits the two properties of (a) Equal Probability and (b) independence (unpredictability), it has been proved that the λ-sequence is a random walk, as $n \to \infty$.[5]

The above establishes RH, by affirming that the asymptotic limit of a is necessarily $1/2$, thereby satisfying (5).

However, in the Main paper we use a more rigorous procedure using Khinchine and Kolmogorov's Law of the Iterated Logarithm (LIL) to arrive at the same result of proving RH (see pp. 169-170 of ECR).

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[3]I had used towers (sets) each containing perfectly ordered, uniformly increasing integers. So the method of mapping is fine and can be justified (as it will not lead to any Cantor-type of paradox). To avoid even this kind of objection (which is misguided anyway), I have given another proof of "Equal Probabilities" that does not use mapping techniques (pp 195-197 of ECR)

[4]See p. 168 of ECR and also my comments on p. 36 which justifies the choice of the relationship: $\lambda(n) = f(\lambda(n-1), \lambda(n-2), \lambda(n-3), ..., \lambda(n-M))$.

[5]In Appendix V, page 174 of ECR, I give an additional 'Theoretical Physicists Proof' that the λ-sequence follows the square root law, which can be read out of curiosity but is unnecessary for the RH proof.

5 Experimental verification

In the last Appendix VI, page 177 of ECR, numerical experiments are described and there it is shown that large sequence of lambdas $\{\lambda(n)\}$ behave 'like' random walks (or equivalently like coin tosses). This empirical verification does *not* constitute a proof— which has been done mathematically as described above — but provides some statistical confirmation consistent with the mathematical proof. Interestingly, the λ-sequence does not show detectably non-Random walk behaviour even at surprisingly low values of N.

Kumar Eswaran
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6 A Personal Note and Open Review

PS: I believe I have answered all the questions[6] that I could envisage being asked by sincere readers, and so do not anticipate posting any more explanations of my work on the RH. Nevertheless, if you still seek some minor clarifications you can contact me at email: kumar.e@gmail.com

NOTE: However, I request that a serious challenge or objection to the proof should be addressed to: Dr. P. Narasimha Reddy, Chairman, Expert Committee (email: nrriemann@sreenidhi.edu.in), along with their full name, address, and institutional affiliation. If the Expert Committee believes the question is serious and warrants a response, it will be answered with the entire correspondence published, along with the name and affiliation of the questioner, to maintain the integrity of the Open Review originally conducted for this work. As spelt out by the Expert Committee in the Preface and Foreword of ECR: Ref[2], they "could not think of a fairer way" than the Open Review system. Given the importance of RH to mathematics, in the interest of transparency it is best that the review process that they initiated is respected.

REFERENCE LIST

PLEASE CLICK ON THE LINKS GIVEN BELOW:

1)The Final and Exhaustive Proof of the Riemann Hypothesis from First Principles.
2) ECR: An Expert Committee's Report on Eswaran's Proof
3) A Brief note on Methodology adopted in the Proof of RH
4) Questions and Answers for my proof of RH

The above Note has answers to 6 questions and is 8 pages long. By mistake I drew a line on the 7th page but the note continues to page 8.

[6]Reference[4] contains some questions that I have answered regarding my proof. This could be consulted. Answers to some simple questions which were posed more than 2 years ago, are available in my Project-log in Researchgate.net