OPEN REVIEWS OF THE PROOF OF THE RIEMANN HYPOTHESIS

OF

KUMAR ESWARAN: AN EXPERT COMMITTEE'S REPORT

BY
DR. P. NARASIMHA REDDY (EDITOR)
JUNE 2021

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June 2021

Sreenidhi Institute of Science and Technology

Yamnampet, Ghatkesar, Hyderabad 501301

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Dr. P. Narasimha Reddy (Editor)

This is a Report on the Open Reviews of the Proof of The Riemann Hypothesis of Kumar Eswaran by the Expert Committee comprising of the following Members:

Members of the committee

1. Dr.T. Ramasami, Ph.D. (Leeds) – Honorable Former Secretary, Ministry of Science and Technology, Govt. of India - **Advisor**

2. Prof. P. Narasimha Reddy, Ph.D. (Kakatiya. Univ.) & M.E. (I.I.Sc. Bangalore): Executive Director, Sreenidhi Institute of Science and Technology **- Chairman and Editor**

3. Prof M. Seetharaman, Ph.D. (Univ. of Madras), Formerly Professor and Chair Department of Theoretical Physics, University of Madras - **Member**

4. Prof. V. Srinivasan, Ph.D. (Univ. of Wisconsin), Formerly Professor and Dean School of Physics, Hyderabad Central University - **Member**

5. Prof K. Srinivasa Rao, Ph.D. (Univ. of Madras) Formerly Senior Professor, Institute of Mathematical Sciences, Chennai - **Member**

6. Prof M. D. Srinivas, Ph.D. (Univ. of Rochester); Senior Fellow, Centre for Policy Studies, Formerly Professor of Theoretical Physics Univ. of Madras - **Member**

7. Prof. Vinayak Eswaran, Ph.D. (Stony Brooke, State Univ. of NY), Dept. of Mechanical and Aerospace Engineering, IIT Hyderabad - **Invited Member**

8. Dr. Adindla Suma, Ph. D (Univ. of Leeds) Associate Professor, Dept. of Computer Science, Sreenidhi Inst. of Science and Tech. - **Convener**

CONTENTS

NOTE: The earlier papers Notes of Dr Kumar Eswaran on RH can be found on his Project page in Reserchgate: https://www.researchgate.net/project/The-Dirichlet-Series-for-the-Liouville-Function-and-the-Riemann-Hypothesis

FOREWORD

It gives me great pleasure to introduce the work of one of our Professors, Dr. Kumar Eswaran, who had joined our Sreenidhi Group in 1999 almost from the time we had started this Institution. He has been a Professor of Computer Science and has been teaching various mathematical related subjects in Computer Science and Information Technology ever since.

Almost 5 years ago he had found a proof of the famous unsolved problem namely “The Riemann Hypothesis”. He had put up his proof on the Web and there were very many downloads (numbering in several thousands), and given several lectures on his methods. These lectures were well received and there were no unresolvable negative comments however, in spite of all this there was a reluctance on the part of the Editors of International Journals to put the paper though a detailed Peer review. Therefore after nearly three years of this stalemated situation, we were advised by various scientists to form an Expert Committee who can look into the proof of Dr. Kumar Eswaran.

So in January 2020 we formulated such an expert committee of Scientists. The Expert Committee whose task was to evaluate Dr. Kumar Eswaran’s proof of the Riemann Hypothesis was in a strange and difficult position. Normally, the correctness of a proof of a mathematical problem of great importance is announced by scholars in mathematics who, after having examined the proof carefully, willingly step forward to announce that the proof is correct. But, despite the fact that Dr. Eswaran’s proof has been on the web for nearly four years and has been sent to some mathematics journals, no one has seriously engaged with it.

So the Expert Committee decided to invite over 1,200 experts (mathematicians and theoretical physicists) across the world to participate in an open review of the proof of Dr. Kumar Eswaran. The review was open in the sense that the referees had to be willing to have their names and institutional affiliations openly revealed, so that nothing is done anonymously, nothing can be said that would not be openly available for all other experts to see. We could not think of a fairer way to get the proposed proof assessed. But the status of the Riemann Hypothesis in mathematics is so great that very few were willing to take the risk of openly venturing an opinion on the correctness or the incorrectness of the proof. Only seven scholars responded. The author of the proof responded to these comments, where warranted. We are publishing the proof, the referees’ comments, and the author’s responses in their entirety, without any redactions.

I can vouch for the fact that, under the circumstances, this expert committee has done due diligence; it has done its utmost to conduct an intellectually rigorous, honest, and fair assessment of the proposed proof. **On the basis of the assessment, this expert committee has concluded that Dr. Kumar Eswaran’s proof of the Riemann Hypothesis is correct.**

Mahhe

Dr. K.T. MAHHE
20th June 20, 2021
Chairman Sreenidhi Group of Institutions

PREFACE (of E-Book)

During the month of January 2020, an Expert Committee was constituted by Sreenidhi Institute of Science and Technology under the advice of ***Dr. T. Ramasami*** to examine the purported proof of the ***Riemann Hypothesis by Dr. Kumar Eswaran***. This step, as explained in an earlier public communication by me, was necessitated by the fact that Dr. Eswaran's purported proof titled ***"The Final and Exhaustive Proof of the Riemann Hypothesis from First Principles"***, was lying extant on the Internet for nearly four years and had received thousands of reads and downloads but had not received the assent of a mathematical Journal to subject it to a detailed peer review.

We, as responsible scientists, believe that a paper on such an important problem as RH should not suffer from a lack of review. All of us had listened to several presentations of Dr. Kumar Eswaran and some of us have studied his work in depth. We believe that the proof merited a critical review by the experts at the very least. Hence, we formed a self-assembled committee of mathematicians and theoretical physicists and appealed to the world's mathematicians and scientists to review the proof ***by a transparent and open review system.*** It was intended that the names and institutional affiliations of the reviewers along with their comments would be posted on a publicly accessible website designed for this purpose. We believed that this open review being a transparent process would deliver a fair assessment of the work and is probably the best method for evaluating a putative proof of a problem of the intellectual challenge of RH. We invited mathematicians and scientists of the world and requested them to review the proposed proof and offer their comments. In this connection, ***more than a thousand invitation letters were sent worldwide by email in the month of February 2020***. In this preface I wish to report the results of the above open review, an exercise which took 15 months.

I had written to more than ***1200 eminent Mathematicians and Scientists*** in my capacity as Chairman of the Expert Committee, inviting all of them to participate in a detailed and Open Review of the work on RH by Dr. Eswaran. However, though the ***downloads from the links provided were several thousand***, only a ***total of seven scientists accepted to participate in this open review*** and out of this were two of our own Committee members who chose to take upon this onerous task and one of them was by Invitation.

The Expert Committee have examined the reviews in detail and their comments are in their Summary below. These are followed by the more detailed unexpurgated reviews of Dr. Eswaran's papers on RH, which are also compiled in this e-book.

Before closing, I would like to record my thanks to all the people who were involved in this evaluation exercise: Firstly, the Reviewers: Prof. Ken Roberts, Prof. S.R.Valluri. Prof. WladislawNarkiewicz, Prof. German Sierra, Prof. M.Seetharaman (Committee Member), Prof. V.Srinivasan (Committee Member) and Prof. Vinayak Eswaran for the great deal of trouble they have taken to undertake their reviews.

My profound thanks to Dr. T. Ramasami (Adviser) and the other Committee members: Prof. K. Srinivasa Rao, Prof. M.D. Srinivas. I also thank Dr. A Suma (the Convener) and the support staff Ms. Padmini Suresh, Mr. Ankusham Satyanarayana and Mr. R. Dinesh.

Finally and most importantly we all thank ***Dr. K.T Mahhe, the Chairman of the Sree Group*** and the management for their unstinted support for the conduction of this exercise.

Prof. P. Narasimha Reddy,
Chairman Expert Committee

SUMMARY OF REVIEWS AND COMMENTS OF THE EXPERT COMMITTEE

The Expert Committee have examined the Reviews of the purported proof of RH by Dr. Kumar Eswaran, there were a total of seven reviewers but two of them have posted joint review these six reviews are summarized using a few operational quotations from their reviews followed by their overall comments.

REVIEW 1

By Prof. Ken Roberts and Prof. SR Valluri. Univ. of Western Ontario, Canada

In this joint review which extended by over 20 pages (pages 65-85) they said as follows: they have said as follows:

"We found Dr. Eswaran's work quite stimulating of mathematical ideas, and believe that his work should be brought to the attention of a wider scholarly audience, That is, the proof (or selected portions of the methodology) should be published. The Riemann Hypothesis has resisted the efforts of many of the best mathematicians for over 120 years,..." (in email to convener page 58).
The committee has appreciated the painstaking work of the reviewers and records its thanks.

REVIEW 2

By Prof. WladislawNarkiewicz, University of Worclaw, Poland, A well- known Polish Number Theorist.

This review is in the form of very detailed technical discussions over emails conducted by Professor WladislawNarkiewicz who had worked through many parts of the paper and asked queries and examined the replies, the discussions extending nearly 60 pages (page 86 -140). The committee commends the painstaking review and discussion which was conducted in the spirit of an open and sincere investigation revealing the many subtleties of RH - we thank him. His entire discussion is given in this report for the benefit of readers and posterity.

In his penultimate email Professor Narkiewicz said that arguments were "heuristic", though he says "I agree that the similarity of the considered sequence of values of the lambda-function with a random walk gives some reasons to believe in the truth of the conjecture"(page 129)

Prof Narkiewicz's final reply ended with this sentence: "I want also to stress that the word "heuristic" has no negative meaning. A lot of work of really great mathematicians has been performed in a heuristic way. This applies not only to old times (Euler, Laplace, the Bernoulli's, ...) but also to recent times"(page 139).

REVIEW 3

By Professor German Sierra, Dept. of Physics University of Madrid, Spain.

This is the only negative review (page 141-143) The Reviewer seems to have believed (erroneously) that Eswaran was trying to prove the randomness of primes and also imputed that he (Eswaran) felt Equal Probabilities is sufficient for the proof. Eswaran, in his reply,(page 144-154) protested that he does not hold to either of these views. Since there was no reference by the Reviewer of more than $3/4^{th}$ of the paper, Eswaran requested that the Reviewer kindly read the rest of the paper for the details of the actual proof. Since the Reviewer did not respond, this Review has necessarily to be treated as incomplete and infructuous.

REVIEW 4

By Prof. M. Seetharaman, Formerly Dept. of Theoretical Physics Univ of Madras

This Reviewer after studying the papers of Kumar Eswaran, was very definitive and said the following; "The author's analysis is exhaustive, unambiguous, and every step in the analysis is explained in great detail and established. The conclusions of the author and his result must therefore be considered proven." (Page 7)

REVIEW 5

By Professor V. Srinivasan, Formerly Professor of Physics and Dean Univ of Hyderabad

Professor Srinivasan reviewed the various steps of the proof saying that "by Judiciously using the properties of the random walk problem", it was shown "that Riemann's Hypothesis is true. There is also a numerical proof given." "I compliment the author for solving the Riemann's Hypothesis."(Page 38-39)

REVIEW 6

By Professor Vinayak Eswaran, Dept of Mechanical and Aerospace Engineering, IIT Hyderabad.

Professor Vinayak, who is Kumar Eswaran's younger brother had taken the trouble to spend the best part of two years to understand and study the background material and understand the proof of RH. Therefore he was invited by the Committee to write a review of the proof. He has submitted a very detailed review that summarized all the arguments of the proof and says that there is no doubt that the Riemann Hypothesis is proved. He also submitted an essay which discusses why the RH was so difficult to prove, as there is perhaps only one way it could have been done (pages 8-35)

Readers not fully familiar with the details of the RH could first read DrVinayak's review, as it gives an excellent overview of Dr. Kumar's proposed proof.

After careful perusal of all the arguments in the proof and the reviews from the experts who have responded, the Committee felt that there are no negative arguments that could technically invalidate the proof and therefore have arrived at the firm conclusion that the proof by Dr. K. Eswaran is both credible and acceptable and that the RH can be considered as proven.

The Committee recommends that the detailed unexpurgated Reviews along with the comments of K. Eswaran and his papers and the findings of this Expert Committee, be compiled in the form of an e-book with a suitable Preface, be made available to the world's scientific community for their perusal and for the historical record.

Members of Technical Committee:

1) Professor M. Seetharaman, 2) Professor K. Srinivasa Rao, 3) Professor V.Srinivasan

4) Professor Vinayak Eswaran (Invited Member).

3rd February 2020

Dear Professor,

Sub : Invitation to Review Putative Proof of the Riemann Hypothesis

I write this letter to you and some other eminent mathematicians and scientists in my capacity as the Head of SNIST and member of a self-assembled committee of scientists for seeking your voluntary cooperation in reviewing a research paper written by Dr Kumar Eswaran.

The said paper "The Final and Exhaustive Proof of the Riemann Hypothesis from First Principles" is available and downloadable using the links provided in the Ref. (1)(cited below) which contains also the author's Extended Abstract Ref.(2), which may be read first.

Proof for the Riemann Hypothesis has remained elusive, as you know, since even attempts by established mathematicians to prove RH have not met with success. Many professional journals are weary of reviewing papers on the subject. As a result, the paper of Dr. Kumar has remained without a formal review by professional journals for three and a half years. A pervasive cynicism that a proof of RH is virtually impossible seems to preclude review of papers on proofs to the RH problem.

Dr. Eswaran's surprisingly short putative proof was uploaded on arXiv and ResearchGate back in September 2016. He has subsequently sought to improve the exposition and add several extensions on his website,producing a final version in 2018 entitled "The Final and Exhaustive Proof of the Riemann Hypothesis from First Principles" (see Ref.[1],opcite.). His attempts to get the paper reviewed for its acceptability have not met with success.

We, as responsible scientists, believe that it is not in the interest of mathematics that a paper on RH problem suffers from reluctance to review. All of us have listened to several presentations of Dr. Kumar and some of us have studied his work in depth.We believe that the proof merits a critical review by the experts at the very least. Hence we have formed a self-assembled committee of mathematicians and theoretical physicists and appeal to some mathematicians and scientists to review the proof.

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Tel: Campus: 08415 200595/96/97

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We propose a transparent and open review system. We invite some interested mathematicians of the world to review and offer their comments. The names and institutional affiliations of the reviewers along with their comments will be posted on a publicly accessible website designed for this purpose.

We believe that this transparent process will deliver a fair assessment of the work. Alleged flaws can be seen by all and debated, which will be in the interest of mathematics. An open review will also elicit thoughtful referee reports with clearly articulated and substantiated objections (if any),which is essential for evaluating a putative proof of a problem of the intellectual challenge in the form of RH.

In fairness to the author, the review procedure would provide him an opportunity to respond to objections and criticisms. He on his part, has put up his ideas and findings in preprints, notes, reports and clarifications on his Research Gate website for the last 3 ½ years and which have been read /downloaded more than 8000 times. He had hoped for a crowd sourcing of professional reviews through research gate. He has not received any serious comment/objection which could scuttle the proof, but he has received a few doubts (which he has amply clarified) and a few congratulatory messages.

What the committee proposes to do at this stage is to play a neutral umpire by inviting eminent mathematicians and scientists to post their reviews openly and set up an intellectual debate on the topic. The members of the committee will not serve as reviewers.

We are inviting you to kindly participate in the objective an open peer review system and give the author a chance to establish his proof to the levels of scientific rigor possible at this time. Given the importance of RH,we sincerely hope that you will agree to participate in the open review.We would be grateful if you could inform me of your willingness or otherwise to participate in the review at your earliest convenience.

The only motivation of our committee to undertake this exercise of open review is to facilitate an honest and honorable peer review process for a paper begging to be reviewed by the peers. As a scientist, in our opinion Dr. Eswaran deserves this support from the mathematical community as an honor system of peers.

Thanking you,
Yours sincerely,

Dr. P. Narasimha Reddy
Executive Director

The next page contains References and clickable Links

ON BEHALF OF COMMITTEE MEMBERS:
1) Prof K. Srinivasa Rao (formerly Senior Professor, Institute of Mathematical Sciences, Chennai
2) Prof M. Seetharaman, (formerly Professor and Chair Dept. of Theoretical Physics, Univ. of Madras)
3) Prof. V. Srinivasan, (formerly Professor and Dean School of Physics, Hyderabad Central Univ.)
4) Prof M.D. Srinivasan, (Fellow Institute of Policy Studies and Former Professor Univ. of Madras)
5) Prof P. Narasimha Reddy (Executive Director, Sreenidhi Institute of Science & Tech, JNTUniv. Hyderabad)

ADVISOR:
1) Dr. Thirumalachari Ramasamy , (Former Secretary to Government of India, Ministry of Science and Technology)

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REFERENCES

The MAIN PAPER is contained in REF (1) below; however it is advisable to read Ref(2) first.

1) K. Eswaran: "The Final and Exhaustive Proof of the Riemann Hypothesis from First Principles
https://www.researchgate.net/publication/325035649_The_Final_and_Exhaustive_Proof_of_the_Riemann_Hypothesis_from_First_Principles

2) K. Eswaran: A Pathway to the Riemann Hypothesis Preprint March 2019
https://www.researchgate.net/publication/331889126_The_Pathway_to_the_Riemann_Hypothesis

3) A Simple Proof That Even and Odd Numbers of Prime Factors Occur with Equal Probabilities in the Factorization of Integershttps://www.researchgate.net/publication/324828748_A_Simple_Proof_That_Even_and_Odd_Numbers_of_Prime_Factors_Occur_with_Equal_Probabilities_in_the_Factor-ization_of_Integers_and_its_Implications_for_the_Riemann_Hypothesis

4)Project Notes These contain work on the RH project, previous preprints, explanations and clarifications of the proof and replies to questions and technical notes.
https://www.researchgate.net/project/The-Dirichlet-Series-for-the-Liouville-Function-and-the-Riemann-Hypothesis

5) The above website contains the PPT of several Invited lectures on RH. The Lecture delivered in IIT Madras is available here:
Invited Lecture on the Proof of RH IIT Madras, May 2019
https://www.researchgate.net/publication/333185552_Invited_Lecture_at_IIT_MADRAS_On_the_Pathway_to_the_Proof_of_the_Riemann_Hypothesis

LATEST DEVELOPMENT

Meanwhile, there has been a seven lecture series on Kumar Eswaran's proof which was prepared by Prof Vinayak Eswaran , (Professor in IIT, Hyderabadand his brother), out of the latter's own volition and interest. All Vinayak's seven lectures which are at the level of undergraduate STEM students are available in:
VINAYAK ESWARAN :Seven Lectures on Kumar Eswaran's Proof of the Riemann Hypothesis -Consolidated
https://www.researchgate.net/publication/336899740_7_Lectures_on_Kumar_Eswaran%27s_proof_of_Reimann_Hypothesis-Consolidated
The Youtube version of his lectures are in: YouTube Lectures by Prof Vinayakhttps://www.youtube.com/playlist?list=PLRsxymPrOKUAk3eXhK9FZdGAPmgGGrhfb&disable_polymer=true

Part A : Report and Reviews on K Eswaran’s paper and his replies

---------- Forwarded message ---------
From: **seetharaman mahadeva rao** <msr4777@gmail.com>
Date: Sat, Jan 2, 2021 at 11:18 AM
Subject: Report on Dr Kumar Eswaran's paper
To: NR RIEMANN <nrriemann@sreenidhi.edu.in>

Dear Dr. Suma,

Please find attached my report on Riemann Hypothesis by Dr. Kumar Eswaran. My best wishes to him for getting due recognition for this important work.

Best regards,
M Seetharaman

RH_Report.pdf
69K

A report on the work of Dr. Kumar Eswaran on Riemann Hypothesis (RH)

The author has tackled a very challenging problem (RH) that has eluded a fully satisfactory solution thus far.

Instead of dealing with the zeros of the the zeta function $\zeta(s)$ directly, he chooses to consider the inverse function $F(s) = \zeta(2s)/\zeta(s)$ for complex s, which is analytic for $\mathcal{R}(s) > 1$ and whose poles exactly correspond to the non-trivial zeros of $\zeta(s)$. He then applies the powerful complex variable theory and analytic continuation to explore the analytic properties of $F(s)$ to the left of the region $\mathcal{R}(s) < 1$. Clearly RH is proved if it is shown that all poles of $F(s)$ lie on the line $\mathcal{R}(s) = 1/2$. The paper has been devoted to essentially establishing this. The crucial part of the analysis is in showing $L(N) < CN^{a+\epsilon}$ for large N when $1/2 \leq a < 1$, which will make $F(s)$ analytic in the region $a < \mathcal{R}(s)$ and then actually determining that $a = 1/2$. This establishes RH.

The author's analysis is exhaustive, unambiguous, and every step in the analysis is explained in great detail and established. The conclusions of the author and his result must therefore be considered proven.

M Seetharaman
Professor (Retd.)
Department of Theoretical Physics
University of Madras

"The Final And Exhaustive Proof Of The Riemann Hypothesis From First Principles",

by Kumar Eswaran[1]

A Review and Comment[1] by Vinayak Eswaran
Professor of Mechanical and Aerospace Engineering, IIT Hyderabad

Abstract

I believe that Kumar Eswaran has proved the Riemann Hypothesis. Supported at one end by Littlewood's RH Equivalent Statement (1912) and on the other by the Law of the Iterated Logarithm of Khinchin (1924) and Kolmogorov (1929), the proposed proof spans the distance between these two century-old classic results in three steps, each of which can be understood with undergraduate mathematics, as I show in this review.

Preamble

I am gratified to have received a request by the the Committee chaired by Dr P. Narasimha Reddy and advised by Prof T. Ramasami[2] to review Kumar Eswaran's proposed proof of the Riemann Hypothesis (RH). However, as I am an engineer by training, and am also Kumar's brother, I think some justification is necessary as to why I should be reviewing a paper on the most famous problem in Pure Mathematics. So, in what follows below, I will include an explanation as to why I feel qualified to comment on Kumar Eswaran's proposed Proof.

My own education (PhD in Mechanical Engineering, Stony Brook, 1986) and research interest thereafter in computational fluid mechanics exposed me to a fairly high level of mathematics of a purely practical engineering variety (Partial Differential Equations, Linear Algebra, Probability theory, etc). But, until just a few years ago, the only thing I knew of the Riemann Hypothesis was that it was a very important problem in Number Theory that involved a lot of Complex Analysis, which too I gleaned from casual conversations with friends and colleagues who were mathematicians, rather than by direct study. That changed one day in 2016 when Kumar told me that he had had a significant break-through on the RH (which was what became the "Towers" proof of the Equal Probabilities result I will discuss below). I read and understood the proof, perhaps not quite fully appreciating it or its true significance, nor its context in what I had hitherto believed was essentially a problem in Complex Analysis. I was told it was relevant to an Equivalent Statement of the RH.

Still with no idea where it came from or how it was derived, I learnt that statement was : *The Riemann Hypothesis*[3] *is equivalent to the condition that for every* $\epsilon > 0$

$$\lim_{N\to\infty} \frac{\Sigma_{n=1}^{N}\lambda(n)}{N^{\frac{1}{2}+\epsilon}} = 0, \quad \epsilon > 0 \tag{1}$$

[1]The formal review is upto page 15, followed by my informal comment on the proof and its reception.

[2] ex-Secretary for the Department of Science and Technology (DST) India (a position equivalent to the Head of NSF in the USA). https://en.wikipedia.org/wiki/Thirumalachari_Ramasami

[3]RH statement: *The non-trivial zeros of the zeta function* $\zeta(s)$ *all lie on the* $Re(s) = \frac{1}{2}$ *line.*

where the Liouville integer function is defined as $\lambda(n) = (-1)^{\Omega(n)}$ and $\Omega(n)$ is the number of prime factors, multiplicity included, in the integer n. Briefly, $\lambda(n)$ is $+1$ if n has an even number of prime factors, and -1 if it has an odd number (which includes primes), and by definition $\lambda(1) = 1$. For convenience, we will use the *Liouville series* for the *summatory* Liouville function, defined as $L(N) \equiv \sum_{n=1}^{N} \lambda(n)$

Kumar's interest in that Equivalence Statement was triggered by a comment he read[4] in a monograph [6] that contained a compendium of known results on the Riemann Hypothesis. Among them was the statement above which was verbally interpreted by the authors as: "...the Riemann Hypothesis is equivalent to the statement that an integer has an equal probability of having an odd number or an even number of prime factors". In an inspired hour, sometime between midnight and dawn, Kumar had come up with the "Towers" proof of Equal Probabilities.

It soon became obvious that that mere proof of Equal Probabilities was not enough, because of the denominator $N^{\frac{1}{2}+\epsilon}$ appearing[5] in the statement. However, the peculiar form of that denominator, with its $\frac{1}{2}$ exponent, and the numerator comprising a sum of $+1$'s and -1's reminded Kumar of an old paper [8] of S.Chandrasekar, the astrophysicist, on random walks and random flights that he had read (as, it turned out, had I) and Kumar had his second inspiration: *If the Liouville series is a random walk, the RH is true.* At first blush, this looked impossible. The λ's in the sequence are so rigidly deterministic, prescribed by the formula $\lambda(n) = (-1)^{\Omega(n)}$, where $\Omega(n)$ is the number of prime factors in n, that there is not the slightest hint suggestive of *randomness* in the series. Apart from sheer formulaic resemblance, there is little to suggest that the Liouville series could be a random walk.

However, the formal random walk requirements are merely these two (1) that the probabilities of λ's of ± 1 be equal, and that (2) the consecutive λ's be independent of each other. The first seemed to be true, but the second seemed false for the Liouville series. However, there was a possible escape from the latter conclusion: the truth or falsity of the RH would be determined by the $\lambda(n)$'s as $n \to \infty$. Was it possible that the λ's could *become* independent in that limit? But how can you show that?

The solution emerged when Kumar began investigating the origins of the Equivalent Statement (1). It was founded on an analysis published by Littlewood[2] in 1912, and reproduced in Edwards [4]. Littlewood started with the simple idea that if the zeta function had a *zero* at any point s on the complex plane, i.e., $\zeta(s) = 0$, then its reciprocal, $1/\zeta(s)$, would have a *pole* at that point. By using the original infinite series definition of the zeta function and the Euler product formula, he obtained $1/\zeta(s)$ as

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}, \quad Re(s) > 1 \tag{2}$$

[4] Throughout his life since I can remember, Kumar has read advanced mathematical tracts for pleasure, the way other people read novels.

[5] For equal probabilities alone, that denominator would only need have been N, as we shall see later.

where the $\mu(n)$ is called the *Möbius* function and is defined as:

$$\mu(n) = \begin{cases} 1 & \text{if } n \text{ has an even number of prime factors without multiplicity} \\ -1 & \text{if } n \text{ has an odd number of prime factors without multiplicity} \\ 0 & \text{if } n \text{ is neither, i.e., it has some prime factor(s) with multiplicity} \end{cases}$$

It was known that the non-trivial zeros of the zeta function were in the "critical strip", $0 < Re(s) < 1$, as complex conjugate pairs symmetrically placed around the line $Re(s) = \frac{1}{2}$. The equation (2) is valid only for $Re(s) > 1$ but Littlewood proposed to analytically continue it into the critical strip. Of course, such continuation would fail at a non-analytic singularity, like a pole, so *if* $1/\zeta(s)$ *can be continued though* $\frac{1}{2} < Re(s) \leq 1$, *proving there were no poles there [and, by symmetry, in* $0 < Re(s) < \frac{1}{2}$*], the Riemann Hypothesis would be proved.*

Littlewood did not prove the RH but did obtain the following Equivalent Statement to the RH through his analysis:

$$\lim_{N\to\infty} \frac{\Sigma_{n=1}^{N}\mu(n)}{N^{\frac{1}{2}+\epsilon}} = 0, \quad \epsilon > 0 \tag{3}$$

where the *Möbius* integer function $\mu(n)$ is defined above. The similarity of (1) with (3) is obvious. Indeed, the Equivalent Statement (1) is obtained in exactly the same way as (3), by replacing $1/\zeta(s)$ by the function $F(s) \equiv \frac{\zeta(2s)}{\zeta(s)}$ in Littlewood's analysis. Like $1/\zeta(s)$, $F(s)$ too has a pole at every point in $0 < Re(s) < 1$ where $\zeta(s)$ has a zero[6], and if continuable through $\frac{1}{2} < Re(s) < 1$ would prove the RH.[7]

Kumar did not find the derivation of (1) in Edwards[4], and did it himself. First he needed to obtain the canonical form of $F(s)$ to be analytically continued by Littlewood's method:

$$F(s) \equiv \frac{\zeta(2s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}\ , \ \ Re(s) > 1 \tag{4}$$

the similarity of which with (2) engenders the already-mentioned similarity of (1) with (3). However, most unexpectedly, it was the *derivation* of (4) that contained the critical clue that led to the (first) independence proof of the λ's that Kumar was searching for, which I will outline further on in this review.

Kumar then came up with another proof of the independence of the λ's of the Liouville series by showing that any dependence relationship of the type $\lambda_n = f(\lambda_{n-1}, \lambda_{n-2}, \lambda_{n-3}, ..., \lambda_{n-M})$, is impossible for finite M (I will also summarise the argument later in this review). It was also discovered that Littlewood's result combined with the Prime Number Theorem gave

[6] It has an extra pole at $Re(s) = \frac{1}{2}$, which is irrelevant to our purpose as it is outside the $\frac{1}{2} < Re(s) < 1$ region of continuation.

[7] It is uncertain who first suggested $F(s)$ be studied. But the derivation of its Equivalent statement is exactly analogous to the derivation by Littlewood for $1/\zeta(s)$, so the Equivalent statement (1) is here attributed to Littlewood.

an alternate proof of the Equal Probabilities result. So now he had *two* proofs each for the Equal Probabilities and Independence results, and thus the $L(N)$ series was shown to be a random walk as $n \to \infty$.

It is easy to show [9] that the RH equivalent statement (1) is unaffected if any finite number N_0 of leading λs are dropped from the series, and $n = 1, 2, ...$ is then counted from the $(N_0+1)^{th}$. This means that the RH is affected only by the behaviour of $L(N)$ as $n \to \infty$, where it has been proved to be a random walk. What does that mean for the RH?

The first and most obvious result is that the summatory Liouville function $L(N)$ scaled by $\sqrt{N}$, becomes a standard Gaussian variable, in the limit of large N. That is, $\xi \equiv L(N)/\sqrt{N}$ will have a Probability Density Function $P(\xi)$:

$$P(\xi) = \frac{1}{\sqrt{2}}\, e^{-\frac{\xi^2}{2}} \tag{5}$$

that seems to probabilistically 'ensure' that ξ would have low numerical values. Thus, for $N \to \infty$, the value of $\frac{L(N)}{N^{\frac{1}{2}+\epsilon}} = \frac{\xi}{N^\epsilon} \to 0$ for any $\epsilon > 0$ thereby satisfying the RH Equivalent Statement (1). However, without more precise information as to *how* (1) would be satisfied, i.e., with precise bounds, this argument would remain essentially heuristic, unsatisfying as a rigorous proof of the Riemann Hypothesis.

Kumar was alerted by a friend of mine, Dr S.V.Ramanan, a physicist-turned-biologist currently residing in Chennai who had read an early draft of Kumar's paper, that he should look at the possible relevance of a great result in Probability Theory, but little-known outside that field, called the *Law of the Iterated Logarithm* (LIL). Kumar did look at it, and within a few hours realised that the LIL[10] put his proposed proof on a firm basis, as a *true* proof[8], not just a heuristic one.

As Kumar did all this, I followed his every step, as we live in the same city. I could follow much of Kumar's reasoning, as I have a long acquaintance with random walk models, albeit in a very different application. Bookended on one side by a long-accepted Equivalent Statement that appears in standard texts ([4], [5], [6]) on the RH, and on the other by a seminal result obtained by two great pioneers of modern probability theory, there was little in Kumar's own contribution that seemed to be out of the ken of an undergraduate in Science or Engineering. In fact, to confirm to myself that Kumar's proof was tenable, I spent the better part of a full year to understand the mathematical context of the Riemann Hypothesis and the arguments of Kumar's proposed proof and, to test my own understanding, wrote a 7-lecture exposition of the proof that would be comprehended by an undergraduate. The lectures contain the *full proof*, including the origin of and relevant historical developments on the RH. While it does not cover some alternate proofs offered in Kumar's original work, and the empirical statistical evidence he gives on the "$L(N)$-series-is-a-random-walk" idea, I believe that my lectures [7] are the most readable version of Kumar's proof. I recommend them for your perusal!

[8] I suppose there are mathematicians who would still argue whether a *probabilistic* proof is a *true* one, but I will let them argue that matter with the ghost of Kolmogorov!

With this contextual background, let me now make an assertion that otherwise would seem absurd for a person not an expert in Number Theory and Complex Analysis with a particular specialisation in the RH to make: *I fully believe Kumar Eswaran has proved the Riemann Hypothesis.* My only caveat would have been that the two pillars on which the proof is constructed, namely, the Equivalent Statement of Littlewood and the "Law of the Iterated Logarithm" of Khinchin and Kolmogorov, particularly the latter, are outside the zone of my competence. But these have both been accepted for over a century, and their statements are easy enough to interpret and apply, and I am sure are applied correctly in Kumar's proof. Kumar's part of the proof can be broken down into three steps, all dealing with aspects of Random Walks, a subject I have had acquaintance with for several decades. And each step has been proved in more than one way, always accompanied by transparent and compelling intuitional reasoning. As for the rest, there is *nothing in Kumar's own contribution that is outside the scope of undergraduate mathematics.* Hence my confidence that Kumar's proof of the Riemann Hypothesis is correct.

Outline and review of the proposed proof

I will devote the rest of my review to outline the various aspects of the proof, to show that that they can indeed be understood by an undergraduate. As mentioned, there are three steps of Kumar's proof: The first two consist of showing that the summatory Liouville function $L(N)$ is a random walk (of unit variance and "time-step"), as $N \to \infty$, by showing that (a) the probabilities are equal that the $\lambda(n)$'s equal ± 1, and that (b) $\lambda(n)$ is independent of the λ's of the numbers preceding n by any finite distance. The third step (c) is to show that $L(N)$ being a random walk, as $N \to \infty$, is sufficient to prove the Equivalent Statement (1) and thus the Riemann Hypothesis. This requires the application of the LIL of Khinchin and Kolmogorov. What follows now is largely taken my Lectures [7], which can be referred for the details.

Proof of equal probabilities

The first step in Kumar Eswaran's proposed proof of the RH is that the λ's of the summatory Liouville function $L(N) \equiv \sum_{n=1}^{N} \lambda(n)$ have equal probabilities of being ± 1, in the limit of N going to infinity.

1. The Towers proof

In his original paper [1] Kumar did this by the "Towers proof" wherein he reorganised the natural numbers into a system of infinite subsets such that every natural number (except 1) is systematically, i.e., algorithmically, placed in one and only one of these subsets. These subsets are such that their members are ordered strictly by size, with each member with $\lambda = 1$ being followed by another with $\lambda = -1$, and vice-versa. Such a system is not unique, as Kumar later proposed another system [11].

Let us consider the system from the original paper. For every natural number n, other than 1, we can do this: factorise the number and represent it as $n = mP^L$, where P is its largest prime factor, with multiplicity L, and m is product of all the factors (including multiples) smaller than P (if any, otherwise $m = 1$). From the prime factorisation theorem it follows that this representation, with $(m,\ P,\ L)$, is unique for each n. We then place the number as the L^{th} member of the "Tower", i.e., infinite subset $S_m^P = \{mP^1,\ mP^2,\ mP^3, ...\ mP^L, ...\}$. As each consecutive member has one more prime factor (as we include multiplicities) than the previous one, the λ values of the consecutive members of S_m^P will alternate between $\lambda = \pm 1$. Its members are also ordered by size as each member is larger than the previous one. It is clear that each natural number $n > 1$ will appear in one, and only one, of the subsets S_m^P, and that the system will represent *all* the natural numbers, and each one only once, and so constitutes an alternate representation of the natural number system (except $n = 1$).

The subsets each have a direct correspondence with, say, the natural sequence of odd and even numbers in the number system in that every odd number ($\lambda = -1$, say) is followed by an even number ($\lambda = 1$), and vice-versa, and are ordered strictly by size. So just as we can say that a "randomly chosen" natural number would with equal probability be even or odd, a randomly chosen member of any subset of this system will equally probably have a λ of ± 1. Now considering a "randomly chosen" natural number, we can say that it will be placed in one unique subset of the system, where it will be equally probable to have a λ of ± 1. Thus a "randomly chosen" natural number[9] will be equally probable to have a λ of ± 1. This, in essence, is Kumar's Equal Probabilities proof.[10]

2. Proof by the Prime Number Theorem

However elegant and interesting Kumar's Equal Probabilities proof was in its use of only elementary arithmetic, there is another way to obtain the same result by the way of the Prime Number Theorem (PNT) and Littlewood's result. The PNT was also motivated by Riemann's famous 1859 lecture to the Berlin Academy where he proposed the RH (see [3] for the historical details). In that lecture, Riemann had advanced an exact formula for the *Prime Counting Function* $\pi(n)$ that determines the number of primes less than or equal to n. The leading term in this exact formula (which was proved in 1895 by H. von Mangoldt) is the Logarithmic Integral function $Li(n) = \int_2^n \frac{dx}{log(x)}$ (which "improves" Gauss' estimate $\pi(n) \sim \frac{n}{log(n)}$). [The remaining "error term" involves the zeros of the zeta function and gives the context the Riemann Hypothesis].

H. von Mangoldt showed that the PNT would follow directly from the result that *all*

[9] An interesting aspect of Kumar's proof is that he does not need to specify how such a "random" natural number would be chosen. Such an algorithm may not even exist.

[10] Kumar goes into some detail to show that towers with their *first* λ equal to ± 1 would be equal in number. I do not think that proof is required for the probabilistic interpretation offered here, as the first λ being +1 or -1 is irrelevant to the fact that the λ's in that tower will become equally probable as its members increase to infinity — which is all that is required in the Equal Probabilities proof.

the non-trivial zeros of the zeta function have real part less than 1. The Prime Number Theorem, which is the formal result that $\pi(n) \sim Li(n)$ asymptotically, was proved independently by Hadamard and Poussin in 1896 by showing there are no zeros of the zeta function on the $Re(s) = 1$ line[11] ([3] pp 155-6). Much later, in 1949, Alte Selberg gave an *elementary* proof of the PNT, i.e., without using Complex Analysis.

The above may be used with Littlewood's result to obtain another proof of Equal Probabilities. Littlewood's result (see Lecture 3 of my Lecture Notes) says that the analytic continuation of $F(s)$ backwards from $Re(s) > 1$ will be possible through $a < Re(s) \leq 1$ if and only if

$$\lim_{N\to\infty} \frac{L(N)}{N^{a+\epsilon}} = 0, \quad \epsilon > 0 \tag{6}$$

and that the poles of $F(s)$ corresponding the non-trivial zeros of $\zeta(s)$ would then lie between $1 - a \leq Re(s) \leq a$ by symmetry considerations.

The PNT shows there are no zeros of the zeta function on the $Re(s) = 1$ line, that is, it must be the case that $a < 1$. This means that we can always find an $\epsilon > 0$ such that $a + \epsilon = 1$. So the PNT implies that

$$\lim_{N\to\infty} \frac{L(N)}{N} = \frac{N^+ - N^-}{N} = \frac{N^+}{N} - \frac{N^-}{N} = P_+ - P_- = 0 \tag{7}$$

where N^+ and N^- are the numbers of $+1$'s and -1's in the N λ's in $L(N)$, so $N^+ + N^- = N$ and $L(N) = N^+ - N^-$, and P_+ and P_- are the respective probabilities of $\lambda = \pm 1$, as $N \to \infty$. The PNT thus implies that the two probabilities are equal, as $N \to \infty$, thereby proving the Equal Probabilities result.

As (7) is an equivalent statement of the PNT, *Kumar's Towers proof of Equal Probabilities is an alternate proof of the PNT*. The ordering by size within the infinite subsets used in that proof ensures that the members in each subset are sequentially included in the $L(N)$ series as N increases, with the probabilities of $\lambda = \pm 1$ becoming more exactly equal in each subset as more members are included, ensuring that, as $N \to \infty$, $P_+ = P_-$ in each subset and as a whole in the entire number system. Whether Kumar's proof will constitute an *elementary* proof of the PNT, I will leave for experts to decide. For, while the Towers argument uses only very elementary arithmetic, the connection to the PNT is through Littlewood's result, which *does* use Complex Analysis.

I have spent more space here on an alternate proof, which was not mentioned in Kumar's original papers, than on Kumar's own proof of the Equal Probabilities result only to make the point that the said result is secure for Kumar's approach towards the RH.

[11] which was all that was necessary, as it was already known that there could be no zeros with $Re(s) > 1$.

Proof of the independence of the λ's

The second step in Kumar Eswaran's proposed proof of the RH is that the $\lambda(n)$'s of the summatory Liouville function are independent, as $n \to \infty$, of the λ's of the numbers preceding n by any finite distance.

The first proof

As mentioned above, the genesis of this proof came from the derivation of the canonical series form of $F(s) \equiv \frac{\zeta(2s)}{\zeta(s)}$ that is analytically continued by Littlewood's method to obtain the Equivalent Statement (1).

In the region $Re(s) > 1$, the zeta function has two alternate forms:

$$\zeta(s) \equiv \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{\prod_p (1 - \frac{1}{p^s})}, \; Re(s) > 1 \tag{8}$$

where the second comes from Euler's product formula and involves an infinite product over all primes. The derivation of (4) requires us to start by using this product form

$$F(s) \equiv \frac{\zeta(2s)}{\zeta(s)} = \frac{\prod_p (1 - \frac{1}{p^s})}{\prod_p (1 - \frac{1}{p^{2s}})} = \frac{1}{\prod_p (1 + \frac{1}{p^s})}, \; Re(s) > 1 \tag{9}$$

and then express each term in the last product by the well-known infinite expansion[12] which are then all "multiplied-out" to get the canonical form for $F(s)$ as (4). [The entire derivation is given in my 3rd Lecture on Kumar's proposed proof [7]]. The critical step comes when the powers of the primes are grouped and replaced by the unique natural number that is their product

$$p_1 \times p_2 \times ... \times p_q \times p_m \longrightarrow n$$

where $p_1, p_2, ..$ can be repetitions of the same prime (i.e., multiplicity is allowed). Every natural number appears once and only once in the series, which can therefore be written as an infinite series in n, as is done in (4). But this simple step contains a clue that is of utmost importance to the independence question. If the coefficient in (4) of the term $\frac{1}{p}$ involving the single prime p is $\lambda(p)$ then the coefficient of n, $\lambda(n)$, is simultaneously obtained in this step as

$$\lambda(p_1) \times \lambda(p_2) \times ... \times \lambda(p_q) \times \lambda(p_m) \longrightarrow \lambda(n) \tag{10}$$

from which we obtain the *true*[13] multiplicative property of the λ's as:

$$\lambda(p_1 \times p_2 \times ... \times p_q \times p_m) = \lambda(p_1) \times \lambda(p_2) \times ... \times \lambda(p_q) \times \lambda(p_m) \tag{11}$$

[12] $\frac{1}{1+x} = 1 - x + x^2 - x^3 ... \quad |x| < 1$

[13] which refers both to the *values* of the λ's as well as their *positions* along the natural number sequence

In the infinite expansion and multiplication to obtain the canonical form of $F(s)$, it turns out that the coefficient is -1 for every term $\frac{1}{p}$ with a single prime p. That is, $\lambda(n) = -1$, if n is a prime. Substituting this information into (11) gives us $\lambda(n) = (-1)^{\Omega(n)}$, as it was first defined. But to mechanically make this substitution of $\lambda(p) = -1$ is to lose all the insight contained in (11) which tells us not just the *value* of $\lambda(n)$ but also where this value will be *placed*, i.e., at location $n = p_1 \times p_2 \times ... \times p_q \times p_m$ in the sequence of the λ's of natural numbers.[14] This is obviously of significance when we recall that *if* the Liouville sequence is to be a Random Walk the λ's must be independent of those preceding them in *that* sequence.

Equation (11) tells us the *dependence* relationship[15] of the λ's in the Liouville series, i.e., the λ in the n^{th} location of the series is dependent on the λ's in the p_1^{th}, p_2^{th}..., and p_m^{th} locations (where the p_i's are the factors of n) and on *nothing else*. This implies[16] (i) that $\lambda(n)$ is *independent* of the λ of any prime number that is not among its factors, and further that (ii) λ's of any two numbers m and n that are co-primes, i.e., without common prime factors, will be independent, and also that (iii) the λ's of two numbers m and n that *do* share common prime factors but have a greatest common factor smaller than either one,[17] will also be independent.[18]

This says the only possible dependence between m and n $(m < n)$ will occur when n is a multiple of m, i.e, when $n = m \times q$, where q is an integer factor. To be dependent on m, as the smallest possible factor is 2, n needs to be at least twice as large as m. This means $\lambda(n)$ cannot be dependent on the λ of any number between $n/2$ and n, i.e., it is *independent* of the (almost) $n/2$ λ's preceding it in the Liouville sequence. Then, as $n \to \infty$, $\lambda(n)$ *is independent of any finite number of preceding* λ*'s.* This completes Kumar's first proof of independence of the λ's of the Liouville series (as $n \to \infty$).

[14] A useful alternate reading would be: *In the sequence of the λ's of the natural numbers, the λ at location n will be -1 if n is a prime, otherwise, its λ will be the product of the λ's of the prime factors of n.* For this proof it is important to think of $\lambda(n)$ as a discrete binary variable determined by its *location* n along the sequence of λ's on the natural number line.

[15] The dependence of the λ's is merely a reflection of the multiplicative dependence of the natural numbers, all of which can be obtained by the multiplication of the primes – which therefore are the only *independent* numbers, upon which the composite numbers are *dependent*.

[16] Henceforth, $\lambda(n)$, or λ of n, should be read as the λ *at location* n of the natural number sequence.

[17] meaning, each has some unique prime factor(s) not appearing in the other

[18] Let G be the greatest common factor of m and n (and smaller than both). From the multiplicative property we get $\lambda(m) = \lambda(G) \times \lambda(P)$ and $\lambda(n) = \lambda(G) \times \lambda(Q)$, where P and Q will be co-primes and thus have independent λ's, which will make $\lambda(m)$ and $\lambda(n)$ also independent.

The second proof of independence

It is easy to show [9] that the Equivalent Statement (1) is unaffected if any finite number N_0 of leading λ's are dropped from the series, and $n = 1, 2, ...$ is counted from the $(N_0 + 1)^{th}$ natural number: That is,

$$\lim_{N\to\infty} \frac{\Sigma_{n=1}^{N}\lambda(n+N_0)}{(N)^{\frac{1}{2}+\epsilon}} = \lim_{N\to\infty} \frac{L_{N_0}(N)}{(N)^{\frac{1}{2}+\epsilon}} = 0, \quad \epsilon > 0 \tag{12}$$

constitutes another equivalent statement of the RH, with N_0 being any finite number. So *any* number of leading λ's can be dropped from the Liouville series and (12) would say the same thing as (1) regarding the truth or falsity of the RH, both being RH Equivalent Statements. This means that the truth or falsity of the RH is determined entirely by the behaviour of the λ's "at" infinity[19]. So, the $\lambda(n)$'s that concern us are those at $n \to \infty$.

Further, if, say, $N_0 = 10^{100}$, it is evident that most of the prime factors that directly affect the λ's of the "shifted" series $L_{N_0}(N)$ ($\equiv \Sigma_{n=1}^{N}\lambda(n+N_0)$) will not appear in the series themselves.[20] So there will not be the direct influence on the $\lambda(n)$'s by the constituent primes of n, as shown in (11). However, there could still be the influence of the λ's of non-prime factors of n, i.e., products of some but not all of the prime factors, which could create dependence relations in the L_{N_0} series. So the dependency of $\lambda(n)$ could extend to its composite factors closer in magnitude to n. We now pursue a proof[21] by contradiction, and first assume that such a dependency *does* exist, so that the L_{N_0} series are *not* Random Walks.

Equation (12) actually embodies an infinite number of RH Equivalent statements, for different N_0's, which must *all* be true if the RH is true and *all* false if the RH is false. So if *any* L_{N_0} series is not a Random Walk due to λ dependencies, they all should have similar dependencies.[22] However, the fact that N_0, in (12), could be *any* number means that the sequence number n of the λ's in the shifted series $L_{N_0}(N)$ is effectively decoupled from the natural number $n + N_0$ that actually determines the λ values. Thus any dependencies that exist between the λ's, as $n \to \infty$, would need to be recursive, i.e., refer to the "local" n and its predecessors rather than the "global" natural number $n + N_0$. Further, as $n \to \infty$, any λ dependencies would reach an asymptotic and unchanging functional form, independent of n, connecting the λ's, as seen in (13) below. Kumar thus argues that the only possible dependence relationship would need to be of the form:

$$\lambda_n = f(\lambda_{n-1}, \lambda_{n-2}, \lambda_{n-3}, ..., \lambda_{n-M}) \tag{13}$$

and be operational as n and $n + N_0$ go to infinity. If there is any such relationship, with finite M, then the Liouville series would not be a Random Walk, as the independence condition of the λ's would be violated.

[19] that is, in the endless expanse of numbers extending beyond all reckoning

[20] as the smallest primes appear most frequently as factors of the natural numbers, with 2 appearing in every second number and 3 in every third, while very large primes will be correspondingly less frequent

[21] discussed in more detail in my Lecture 6 [7]

[22] "at" infinity, which alone determines the RH

If the dependence relationship (13) is true then, as n increases toward infinity, each new $\lambda(n)$ in the Liouville series is determined by the previous M λ-values. However, we know that, as the λ's are binary-valued, the argument set of $\lambda_{n-1}, \lambda_{n-2}, \lambda_{n-3}, ..., \lambda_{n-M}$ could have at most 2^M different combinations. So, as n increases, the λ combination in the argument of the dependence function in (12) would necessarily repeat some previously-occurring combination, and yield the same $\lambda(n)$ as before — and then the same $\lambda(n+1)$, $\lambda(n+2)$..., as before, so the series becomes cyclic thereafter. Further, the Equal Probabilities result would ensure that each cycle would necessarily comprise equal numbers of $\lambda = \pm 1$, so the $L(N)$ would return to the same value at the end of each cycle. Therefore, $L(N)$ would remain finite even while $N \to \infty$. If this happens, Littlewood's condition (6) would be satisfied for $a = 0$, implying that there are *no* zeros of the zeta function between $0 < Re(s) < 1$ – which is untrue because we know there *are* zeros[23] on the $Re(s) = 1/2$ line. Hence, the only possible dependency function (13) previously allowed is now shown to be impossible. Thus the independence of the λ's over finite strips is assured as $n \to \infty$, completing Kumar's second proof.

The empirical evidence that $L(N)$ is a Random Walk

It has been shown above that the Liouville series $L(N)$ should become a Random Walk as $N \to \infty$, by proving that $\lambda(n)$ values are equally probably ± 1, and that they are independent of the preceding λ's, and therefore behave like "coin-tosses", at least for high values of n.

That the $L(N)$ series is a Random walk is a proposition that can be empirically tested by statistical means in an exercise to, first, verify if indeed the proposition seems true and, second, to throw some light on what that condition "$N \to \infty$" means in practical terms. The $\lambda(n)$'s of the series themselves can be obtained by brute-force factorisation of n by mathematical software, and then using $\lambda(n) = (-1)^{\Omega(n)}$. Kumar did this in his paper, and the results are interesting.

A standard and widely used test of statistical hypotheses is the Chi-squared (χ^2) test which comprises in computing a diagnostic χ^2 value from the observed empirical data and then checking whether that value falls within the normal range of values that would occur if the null hypothesis were true. The "normal range" is left to the analyst's choice but usually includes 95% of possible χ^2 values. If the empirical χ^2 value falls outside this chosen range, the test is said to have rejected the null hypothesis.

In our case we would like to see if the $\lambda(n)$'s of the $L(N)$ series from, say, $n = n_0$ to $n_0 + M - 1$ are statistically distributed as if they were M coin tosses: this is the null hypothesis. For this case $\chi^2 = \frac{(M^+ - M^-)^2}{M}$ where M, M^+ and M^- are, respectively, the numbers of the λ's in the stated range, and the numbers that are $+1$ and -1. The difference $M^+ - M^-$ can be directly obtained from the $L(N)$ series as $L(n_0 + M - 1) - L(n_0 - 1)$. The difference of the probabilities of $\lambda = \pm 1$ in that range is $\Delta P \equiv P^+ - P^- = \frac{(M^+ - M^-)}{M}$.

[23] an infinite number, as proved by Hardy

For the full series ($n_0 = 1, M = N$), the χ^2 value is $\chi^2 = \frac{L(N)^2}{M}$ which is the square of the proposed "normal gaussian variable" $\frac{L(N)}{\sqrt{M}}$, while $\Delta P = \frac{L(N)}{M}$. The expected χ^2 distribution for this case is exactly Gaussian and $\chi^2 \leq 3.84$ is the 95% bench mark for rejecting the null hypotheses (if the computed χ^2 value exceeds 3.84 more often than 5% of the time).

Kumar did extensive statistical testing of the λ's, over different and consecutive segments, typically with M=1000, and also the whole range, from $n = 1$ to 176 trillion, which can be seen in the Appendix IV of his paper. He shows that over all segments and the whole range the λ's are statistically indistinguishable from coin-tosses in the Chi-squared test, apart from showing ΔP steadily decreases to reach values below 10^{-7} by the end of the range. While it can be argued that $n = 176,000,000,000,000$, is not a high number in the context of the RH, the statistical finding that $L(N)$ is indistinguishable from a Random Walk, right from the start of the series, gives powerful empirical backing for the main line of attack of Kumar's proposed proof.

Why a Random Walk $L(N)$ implies the RH is true

We have seen that, as $N \to \infty$, the Liouville series $L(N)$ satisfy the two conditions[24] required of a Random Walk, i.e., that (1) the probabilities are equal that the λ's of the series are ± 1, and (2) that the $\lambda(n)$'s are independent of the λ's of the numbers preceding n by any finite distance. The remaining question, whether this proves the Riemann Hypothesis, will be taken up in this section.

The Heuristic 'proof'

If the the Liouville series $L(N)$[25] is a Random Walk then, as previously mentioned, $\xi \equiv L(N)/\sqrt{N}$ becomes a standard Gaussian variable as $N \to \infty$, with zero mean and unit standard deviation, which will remain probabilistically confined to low values (with, say, the probability of $|\xi| > 10$ being around 10^{-23}), thus ensuring that the value of $\frac{L(N)}{N^{\frac{1}{2}+\epsilon}} = \frac{\xi}{N^{\epsilon}} \to 0$ for any $\epsilon > 0$ thereby satisfying the RH Equivalent Statement (1).

This idea was the intuitional motivation for Kumar's line of attack on the RH of first showing that the $L(N)$ series is a random walk. However, having done that, the argument put forward above would possibly have been at best considered *heuristic*, as well as one that left important questions related to the RH (for example, the width of the Critical Line) unanswered. A particularly niggling question is that if we choose $\epsilon = 0$ in (1), Littlewood's analysis predicts that $L(N)/\sqrt{N}$ would be unbounded (as the zeros of $\zeta(s)$ and the poles of $F(s)$ would then necessarily be at $Re(s) = \frac{1}{2}$) while the heuristic argument would say that $L(N)/\sqrt{N}$ would still be a Gaussian variable, and therefore very much finite. However, all these questions were put to rest, and the proof was raised

[24] That this has been shown only for $N \to \infty$ does not matter in this context, as the Equivalent Statement (1) is determined entirely by the behaviour of $L(N)$ as N goes to infinity.

[25] or, more correctly $L_{N_0}(N)$ shifted by a sufficiently high N_0

to a high level of rigour by the introduction of the Law of the Iterated Logarithm, as shown below.

Formal proof through the Law of the Iterated Logarithm (LIL)

The LIL was respectively proved and reformulated in the 1920 by two of the great names of modern probability theory, Khinchin and Kolmogorov. It looks at the *maximum deviation* that the Random Walk can make from the probabilistic zero-mean expectation, during its transient evolution. Its importance in this context is that $L(N)$, assumed to be a Random Walk, too would have deviations from the Random Walk zero-mean expectation, as n increases without limit. *If* the maximum deviation obeys a power-law of N with an exponent greater than $\frac{1}{2}$, this would invalidate Equivalent Statement (1) and disprove the Riemann Hypothesis. If, on the other hand, these deviations can be shown to have no greater than an $\frac{1}{2}$ exponent[26], as $N \to \infty$, then (1) would be satisfied and the Riemann Hypothesis proved. It is most fortunate that the LIL can be easily re-stated in terms of (1).

The LIL[27], when applied to the Liouville series (now assumed to be a random walk of zero mean and unit variance) gives the *limit superior* (i.e., maximum value) of $L(N)$ for $N \to \infty$ as $a.s.$[28] being $L_{max}(N) = \sqrt{2}\sqrt{N}\ e^{\frac{1}{2}log\ log\ logN}$, where $log\ log \log N = log(log(log(N)))$ involves the nested ("iterated") use of the natural logarithm.
Substituting this in (1) tells us:

$$\lim_{N\to\infty} \frac{L_{max}(N)}{N^{\frac{1}{2}+\epsilon}} = \frac{\sqrt{2}\ e^{\frac{1}{2}log\ log\ logN}}{N^{\epsilon}} = \frac{\sqrt{2}\ e^{\frac{1}{2}log\ log\ logN}}{e^{\epsilon logN}} = \boxed{\lim_{N\to\infty} \frac{\sqrt{2}}{e^{(\epsilon-d_N)\ logN}} = 0, \quad \epsilon > 0} \tag{14}$$

is necessary for the Equivalent Statement (1) to be satisfied, where $d_N \equiv \frac{log\ log\ logN}{2\ logN}$ is a monotonically decreasing function of N, with limit 0 as $N \to \infty$. We see that no matter how small ϵ (> 0) is, d_N will be smaller beyond some N_ϵ, so that the last term goes to zero, thus satisfying (14) and (1), so proving the Riemann Hypothesis!

Littlewood's result (see Lecture 3 of my Lecture Notes) says that the analytic continuation of $F(s)$ backwards from $Re(s) > 1$ will be possible through $a < Re(s) \leq 1$ if and only if

$$\lim_{N\to\infty} \frac{L(N)}{N^{a+\epsilon}} = 0, \quad \epsilon > 0 \tag{15}$$

[26] which is the lowest value it can have, as the mean magnitude of a Random Walk variable is $O(N^{\frac{1}{2}})$

[27] Traditionally stated as [10]: Let $\{Y_n\}$ be independent, identically distributed random variables with means zero and unit variances. Let $S_n = Y_1 + ... + Y_n$. Then

$$\lim\sup_{n\to\infty} \frac{\pm S_n}{\sqrt{2n\ log\ log\ n}} = 1,\ a.s.$$

[28] $a.s.$ "almost surely", with zero probability of violation. Because of symmetry, we do not need to consider the *limit inferior*, the lowest value.

and that the poles of $F(s)$ corresponding the non-trivial zeros of $\zeta(s)$ would then lie between $1 - a \leq Re(s) \leq a$ by symmetry considerations. As we have shown this for $a = \frac{1}{2}$, the zeros will all lie on $Re(s) = \frac{1}{2}$, thereby also showing that the width of the Critical Line is zero. By putting $\epsilon = 0$ in (14), we see that the LIL gives us $\frac{L_{max}(N)}{\sqrt{N}} = \sqrt{2\ log\ logN}\ a.s.$. That is, the "Gaussian random variable" ξ will "almost surely" go to infinity as $N \to \infty$, which satisfyingly resolves the paradox stated at the end of the last section by agreeing with Littlewood that, indeed, $L(N)/\sqrt{N}$ would be unbounded as $N \to \infty$ as the zeros of $\zeta(s)$ are all at $Re(s) = \frac{1}{2}$. Which concludes the proof.

So there we have it. I have outlined the three steps of Kumar's proposed proof of the RH, starting from Littlewood's Equivalent Statement (1) and ending with the the Law of the Iterated Logarithm (LIL) of Khinchin and Kolmogorov. The first two steps involve showing the λ's of the Liouville series $L(N)$ appearing in (1) are equally probably ± 1, and are independent over finite strips, as $N \to \infty$, thereby effectively becoming "coin-tosses", and making the $L(N)$ series a Random Walk. These steps have both been shown in two different ways, where the first proof in each case is accessible not just to undergraduates but perhaps also to high-school students. The third step, that $L(N)$ being a Random Walk proves the RH Equivalent Statement (1), is shown by the LIL which establishes that Littlewood's Equivalent Statement (1) will be satisfied as $N \to \infty$. Littlewood's method and the LIL are well-established results that are a century old, while Kumar's contribution in connecting them can be understood by undergraduate students. On these facts I rest my conviction that Kumar has proved the Riemann Hypothesis.

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Some personal reflections on the proof and its reception

Having perforce lived with Kumar's work these last several years, and having spent much time to understand both his proposed proof and the context and history of the Riemann Hypothesis, I have some personal thoughts that I would like to share, if I may, both on the RH and on Kumar's experiences on proving it. Yes, I shall assume, in what follows, that Kumar *has* proved the RH – for it seems so obvious to me that he has – even while I recognise that the jury to decide this, far from being out, has not yet even been empaneled, well into the fifth year after Kumar first communicated his proposed proof to the leading journals of mathematics, none of which even bothered to formally review it.

However, I do not want to restrict myself to Kumar's woes, but intend to use my unique position as the first person[29], after Kumar, to know the proof of the RH, and count coup, so to speak, on this famous and enduring problem. So I shall return to Kumar's difficulties after having first unburdened myself of my thoughts[30] on the proof.

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The first question that would strike the mind of someone who, like me, has accepted that Kumar's proof is indeed correct, is why would it take so long to be discovered? All the mathematics required in the proof is at least a century old – a century in which the Riemann Hypothesis was widely regarded as the greatest problem in mathematics, and every talented pure mathematician would have paused, at least for a while, to consider its proof. Some of the greatest mathematicians of the 20^{th} Century did, as is well-known. Why did none of them succeed, particularly if the proof is so simple?

There are several reasons that can be given but, to my mind, the main one is this: *There was, and is, only one way to prove the Riemann Hypothesis.* To borrow imagery from Tolkien, the only way to breach the defences of the Lonely Mountain, which otherwise would resist a frontal assault by large armies, is through a single secret door. That door is the coin-toss behaviour of the Liouville integer function, $\lambda(n)$, and the only existing key to that door is the Liouville series for the function $F(s)$. But again, why is that? Because, I suggest, *the random-walk behaviour of the λ's is the reason why the non-trivial zeros of the zeta function fall on the $Re(s) = 1/2$ line.*

To say that something is the reason for something else implies *causality.* Physics has the so-called arrow of time for distinguishing cause from effect between concomitants. Mathematics has no comparable touchstone, and works through non-temporal logical implications and equivalences alone to create structures of truths consistent with its axioms. So, in mathematics, to say that A *causes* B, needs an explanation. What does it mean then to say that the random-walk of λ's *causes* the zeros of the the zeta function to lie on the $Re(s) = 1/2$ line?

[29] Our brother Mukesh also read Kumar's drafts but, living in the same city, I followed his work closely.

[30] which have been greatly helped by my discussions with Kumar and Mukesh.

While mathematics lacks an arrow of time, it does have a hierarchal structure that serves to establish precedence – the mathematics of various numbers, i.e., natural, integer, rational, irrational, real and complex numbers, are separate logical structures, but each with axioms, elements, and operations that have been derived and generalised from the preceding one, with the natural numbers being first. Each structure rests on the previous ones like the successive floors of a building where the lower storeys were built independently of the upper ones, but not the upper ones of the lower. So it is possible to conceive of a mathematics of natural numbers without considering complex numbers, but not a mathematics of complex numbers without considering natural numbers.

We have seen that the proof of the RH was arrived at by an Equivalent Statement that tied the behaviour of the λ's by an ironclad "if and only if" condition to the zeros of the zeta function. The first is defined entirely by the natural numbers, while the latter is embedded deeply in Complex Analysis. Equivalence is a two-way implication, meaning that both propositions it contains are together true or together false. However, it would be absurd to say that the behaviour of the λ's is determined by the zeros of the zeta function, but perfectly logical to say that the latter is determined by the former — i.e., that the behaviour of the λ's *causes* the zeros of the the zeta function to lie on the $Re(s) = 1/2$ line, as no other condition is required for the latter.

However, even this does not entirely justify my further claim that there is only *one way* to prove the Riemann Hypothesis. Consider then the prime numbers. Primes enter the RH through the Euler product formula definition of the zeta function but are "reabsorbed" into the natural numbers, i.e., lose their explicitly separate identity, in the many equivalent representations of the zeta function. So whatever influence the primes have on the zeta function is implicit, i.e., hidden. While the zeta function contains some information of the primes – the zeros of the zeta function are used to make the Prime Counting function more precise – it cannot also tell us why the zeros are located where they are. That information has to come from elsewhere.

The first place to look, naturally, would be Complex Analysis or, stated more generally, Analysis. Could the peculiar behaviour of the λ's have worked itself into the axioms of Analysis, so that the RH could be proved entirely by Analysis? This seems impossible because Analysis does not, fundamentally, even have the concept of *prime numbers* – as every real or complex number is divisible – and so can say nothing of the λ's, which are entirely determined by the primes. This has a very significant implication — that *the Riemann Hypothesis cannot be proved by Analysis alone*[31] — which would explain why it was so difficult to prove, as Complex Analysis was always the foremost line of attack.

It has sometimes been suggested, usually not very seriously, that the RH could possibly be an example of the correct-but-unprovable-assertions of Godel's First Incompleteness

[31] Kumar informs me that some mathematicians have anticipated this, e.g., J.B. Conrey: "It is my belief that RH is a genuinely arithmetic question that likely will not succumb to methods of analysis." https://www.ams.org/notices/200303/fea-conrey-web.pdf pg 353

theorem, no natural examples of which have been found apart from the one constructed by Godel himself. My argument here is that this is indeed true: The Riemann Hypothesis, which is a statement in Complex Analysis, cannot be proved by Complex Analysis. It required a bridge – provided by Littlewood's analysis — out of Complex Analysis to Elementary Number Theory, and is finally proved by Probability Theory!

Even granting that the location of the zeros of the zeta function are determined entirely by the random-walk behaviour of the λ's, could another way be found to prove the RH other than by using the Equivalent Statement (1)? Is there another key to the secret door into the RH? For this to be possible, whatever else that is used has to precisely reflect the said behaviour of the λ's. One candidate is the original Equivalent Statement 3 of Littlewood, involving the *Möbius* integer function $\mu(n)$ defined in (2).

It is easy to show that *given* the "coin-toss" behaviour of the λ's, i.e, equal probabilities and independence, that (3) indeed would be satisfied[32]. This was actually done by Denjoy [12] in the 1930's. But he did not rigorously *prove* that $\mu(n)$ would have coin-toss behaviour, as Kumar has done with the $\lambda(n)$'s, but instead gave plausibility arguments for the equal probabilities and independence properties. It was not well-received. Edwards ([4], pp 268-269) called Denjoy's argument "quite absurd" and "ludicrous" – and this in a technical monograph on the Riemann Hypothesis – which in retrospect we see was somewhat unfair, as Denjoy had correctly identified the secret door to the RH, even if he did not turn its key — which anyway, being (3), most likely was the wrong one.

While the proof of the Equivalent Statement (3), independently of (1), *may* be possible[33] it is clear that the latter is the larger result, and subsumes the former. So I think that I am right in saying that any proof of the RH would necessarily invoke the random-walk behaviour of the λ's – which is the only key to the secret door into the RH!

So the RH has its basis in that the λ's behave like coin-tosses, and the summatory Liouville function is a random walk. The sheer unexpectedness of this discovery — as both seem to be perfectly *deterministic* and as far from *random* as could be possible — no doubt is one reason why the RH-Equivalent Statement (1), appearing in every modern

[32] The *Möbius* function $\mu(n)$ is equal to $\lambda(n)$ for every n that is prime or a product of single prime factors, and 0 if n has prime factors with multiplicity. So assuming coin-toss behaviour in the λ's (which would carry over to the non-zero μ's due to the independence condition), the *summatory Möbius* function, $M(N) \equiv \sum_{n=1}^{N} \mu(n)$ will still be a random-walk, but for N_p steps, not N, where N_p (for large N, $\pi(N) < N_p < N$) is the total numbers up to N that have single prime factors. So we can say, as with the Liouville function, that $M(N)$ will satisfy:

$$\lim_{N\to\infty} \frac{\Sigma_{n=1}^{N}\mu(n)}{{N_p}^{\frac{1}{2}+\epsilon}} = 0, \quad \epsilon > 0$$

and hence also (3) [by multiplying the above equation by $(\frac{N_p}{N})^{\frac{1}{2}+\epsilon}$].

[33] Kumar's factorisation proof of independence of the λ's will go through to the μ's, but both equal probabilities proofs for the λ's have difficulties, possibly insuperable, of translation to the μ's, as the latter are non-zero only for a subset of the natural numbers, unlike the λ's.

textbook on the RH, was not recognised for what it was — the key to the puzzle of the RH. And so it lay in plain sight, like the purloined letter in Poe's story, to be discovered by an amateur, while professional mathematicians hunted all over the higher domains of mathematics, and even created new ones, for that elusive key.

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The proof of the Riemann Hypothesis is, perhaps, the most anticipated event in mathematics. Number theorists tell us there are hundreds of research papers that assume the RH is true to obtain results that would thereby become theorems the instant it is proved. There are also dozens of equivalent statements, most involving very complex mathematics, that again would be proven true along with the RH. Even without these consequences, the RH itself is seen as the missing corner-stone of a beautiful edifice of Analytical Number Theory that Riemann himself inaugurated. It is also expected that the proof of such a long-standing problem would break fresh ground in mathematics and bring in new ideas that would enliven future research. All this would be true with Kumar's proof.[34]

However a final expectation, that the proof would bring to bear new techniques, is unmet. Let us consider this disappointment. The RH was proved by techniques known for a century, a century in which mathematics made enormous strides. Divorced from the RH, the key finding that the Liouville integer function is a random-walk would have been seen as an interesting but inconsequential result obtained by old techniques, probably insufficient in itself even to guarantee tenure in a so-called great school.[35] Yet, it is most surprising that a proof that uses techniques known since the 1920's defeated Hardy, Littlewood, Hilbert, Selberg and Nash, to name just a few in a much longer list of mathematicians who attempted it. Why?

I have already given the reasons. The RH, although formulated in Analysis, would be impervious to Analysis for its resolution. Analysis would have, of course, been the natural and primary mode of attack used by any 20^{th} Century mathematician who attempted a proof. But that would have been like the drunk's searching under the street light – because there was more light there – for the door key that was lost elsewhere. That search was doomed to be fruitless.

However, even someone with an idea that the answer lay in Arithmetic, the true habitat of prime numbers, would be lost searching for a needle in the proverbial hay-stack, without even an idea what a needle looks like. Who would think that the behaviour of the λ's – entities that have significance only in Analysis, not Arithmetic – would provide the ultimate resolution to the puzzle of the RH? It would have to be a fortuitous event,

[34]Kumar informs me that the insights from his proof would be immediately applicable to certain Dirichlet L-functions in the Generalised Riemann Hypothesis (GRH).

[35] while, *with* the RH considered, it would inevitably confer on its discoverer – given, of course, previous attendance and current employment in a great school and the right pedigree – the Abel Prize, and if young enough, the Fields Medal and *then* the Abel Prize!

somewhat like stumbling onto a secret door during a continent-wide search. But without its key the significance of the door would never have been known. It is most fortunate that a key to that door even *existed*, for — being forged by the unlikely-looking function $\frac{\zeta(2s)}{\zeta(s)}$ — it quite easily may *not* have, making the RH forever out of reach of proof, driving gifted mathematicians to insanity a thousand years from now. And the only hint of the existence of that door and that key was the one-half exponent in the denominator of the expression (1) that, like a sword stuck in an anvil, attracted the attention of a passerby who intuited its meaning in an epiphany.[36]

So the proof has lessons, if not in techniques, for the broader endeavour of mathematics. Riemann's creation of Analytical Number Theory was hailed as a great advance because it allowed the power and wider range of Analysis to be brought to the ancient field of number theory. As mentioned, mathematics comprises several logical systems, for the natural, integer, rational, irrational, real and complex numbers, stacked like the storeys of a building, with each level gaining, through generalisation, greater power and scope. So to bring the methods of complex and real analysis, from several levels above, into play in number theory, the mathematics of natural numbers, was seen as an unqualified good thing. What was, perhaps, not given as much thought is the possibility that something could be lost in that process of generalisation — in this case, the immediacy of the primes — so that the peculiar behaviour of the λ's, which we have seen can be understood by undergraduates, climbs up five levels and is so transformed in that process that it becomes the greatest problem in mathematics.

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However, to say that the path to the proof of the RH was a very narrow and improbable one does not imply that the result itself is inconsequential. The great beauty of the Riemann Hypothesis lies in the connection that it makes between the natural numbers, which surely must have been known to humans at least a hundred thousand years ago, and the theory of complex variables, which is based on an unimaginable abstraction, the square root of a negative number, that was discovered by the human imagination less than three centuries ago. That connection is magnificent, like a bridge traversing different worlds.

It is also a test of that bridge. An engineer, by attaching a sound source to one end of an aircraft wing and a sensor to the other, can discern merely by using high-frequency ultrasonic waves whether the wing is structurally sound. So too the Riemann Hypothesis is a test of the entire structure of mathematics that had its origins in the natural number system and reached an apogee, around the time of Riemann himself, in the theory of complex variables.

[36] If there is a divinity that presides over Mathematics, it surely must be a goddess – as Ramanujan believed, and whom he credited for his inspirations. I like to imagine that the Goddess looked down from the high heavens and saw a seventy-year-old man reading an abstract tract on the Riemann Hypothesis late into a moonlit night, by the last light of Durin's day, and said to herself, "Why, he loves me truly", and impetuously slipped him the intuition of the secret door into the Lonely Mountain, a boon she had denied the greatest mathematicians of the 20^{th} Century...

More than any other field of human endeavour, mathematics is a product of pure thought, and its structural integrity lies entirely in the soundness of the logic underlying that thought. As mathematics progresses by generalisation, its new fields are created by adapting the operations used in the older ones to apply to newer conceptual creations. So the ideas of addition, subtraction, multiplication and division, learnt on natural numbers, are extended and generalised to fractions and negative numbers, and to the so-called real numbers and then to complex numbers. The edifice is created layer by layer, with each stage resting on the previous one. But the new discoveries and theorems almost always pertain to the new field and rarely say anything novel and interesting about the older ones. For example, non-Euclidean geometry, created in the 19^{th} and early 20^{th} Centuries, added not a single new theorem to Euclidean geometry[37], created two millennia ago.

So the newer fields presume the truth of the older ones on which they are based, but they do not discover *new* truths in those older fields. This leaves open a door to a chilling doubt — is "higher" mathematics saying something different from the "lower" even when it is speaking of the latter? The whole of mathematics was constructed so that this should *not* be true. But could an insidious fallacy have crept in during its expansion and generalisation, so that mathematics is no longer consistent?

In 1931, Gödel with his famous Incompleteness theorems, first raised the possibility that axiomatic systems in mathematics — which could be the whole field or any of its branches — may have some propositions that may not be either provable or disprovable. So mathematics is possibly *incomplete*. His second theorem went even further, and said that no such logical system can ensure its own *consistency* — that is, guarantee that its theorems would never contradict one another. Such a contradiction would, of course, be disastrous and invalidate the entire system. Incompleteness one can perhaps live with, but inconsistency is another matter — and far more serious. For it would mean that mathematics of some level may say something *wrong* about another more basic or foundational level. That, surely, is something that must keep the Spirit of Mathematics awake on bad nights. Which is why even mathematics needs a system check as much as an aircraft does. The Riemann hypothesis is just such a check.

But there have been others before. In 1734, Euler solved the so-called Basel problem (see, e.g., [3]), that had been proposed almost a century earlier, of theoretically determining the sum of the infinite series[38]:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \equiv 1 + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{n^2} + ...$$

that had resisted the attack of many well-known mathematicians of the day. Euler's answer — that the sum was $\frac{\pi^2}{6}$ — was particularly stunning for it involved the transcendental number π that had hitherto been known only as the ratio of the circumference of a circle and its diameter, thus exposing an unexpected connection between the natural

[37] although it did create a useful debate on the foundational axioms of Euclidean geometry

[38] which turns out to be related to the zeta function — being precisely $\zeta(2)$

numbers and geometry, and seeming to hint that mathematics is less a human *creation* than a human *discovery*. Most importantly, the result was verifiable, for the series converges quite rapidly and the actual calculation of the first few dozen terms shows their sum approaching ever closer to the value given by Euler for the infinite sum. This gave confidence[39] that the "higher" mathematics used to derive the result — already far removed from the counting-of-natural-numbers origins of arithmetic — was sound.

The Riemann Hypothesis is the Basel problem on steroids, as they say. It involves every significant step in mathematics leading from the natural numbers to what is called Analysis — which had reached a very high level of achievement just around Riemann's era. To elaborate, the Riemann Hypothesis connects the entire sequence of mental leaps from whole numbers, addition, subtraction, multiplication, division, fractions, negative numbers, real numbers, imaginary numbers, real functions, infinite series, limits, continuity, differentiation, integration, complex function theory, analyticity, analytic continuation, convergent and divergent series and then completes the circle by referring back to the natural numbers. The end of all that exploration is to arrive where we started, as the poet said.

So what finally matters with the Riemann Hypothesis may not be what it *precisely* says about the natural number system but that is says something *true* — thereby validating to some degree the structure of mathematical analysis that was created over hundreds of years. This is important. For more than any other branch of mathematics, modern science and technology depends on Analysis. Analysis allows aeroplanes to fly, radios to hum, skyscrapers to stand, and $E = mc^2$ to be written. It is no doubt this intuitive understanding of its importance that drove Hilbert, Hardy, Littlewood, and scores of other great analysts to give the Riemann Hypothesis its supreme place in mathematics — its proof, to recall Kronecker's remark, would justify the work of man before God, so to speak. Therein may lie its true significance.

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Having considered what the Riemann Hypothesis could mean to mathematics in general, let us consider some of the specifics shown up by Kumar's proof.

Fittingly, for a result that has obsessed number theorists for a century, the Riemann Hypothesis includes the *entire* domain of natural numbers. Its truth depends on the result of a head-count made by the summatory Liouville function, $L(N)$, comprising the sum of the λ's of *every* natural number, wherein the λ of 5, say, counts equally with that of five thousand, or five trillion, or that of a 5 trillion *digit* number. Yet, to paraphrase Kipling's injunction, all numbers count with it but none too much: for, as we have seen, we can drop any finite number of λ's from the count, or even an infinite number — say, that of every alternate number — and yet the truth of the Riemann Hypothesis would

[39] As it turned out, prematurely, for part of the method used by Euler could be justified by mathematicians only a century later. So the empirical verification turned out to give a premature commendation of the logic used.

be unwavering. The RH is truly the will of all natural numbers, freely and independently exercised[40], and so the most *democratic* result in all of mathematics!

And it takes an extraordinarily large "poll" to even indicate the final result of that collective will. We have seen, in equation (14), the function $d_N \equiv \frac{log\ log\ logN}{2\ logN}$ represents the deviation from the critical line, $Re(s) = \frac{1}{2}$, possible when the summatory Liouville function is counted to some finite N. That is, it represents the "half-width" of the critical line for finite N, going to 0 as $N \to \infty$. Now d_N, surely, is the slowest decreasing function of any significance in mathematics. For $N = 10^{100}$, i.e. one googol, vastly more than all the atoms in the Universe, $d_N = 3.7 \times 10^{-3}$. For $N = 10^{1000000}$, $d_N = 5.7 \times 10^{-7}$, less precision than that of a pocket-calculator. The RH truly spans infinity.

In the 20^{th} Century, rising with digital computing, Monte Carlo techniques – numerical simulations of stochastic processes ranging from solutions of Partial Differential Equations to variations of the Stock Market – required the generation of large numbers of random numbers distributed in specific ways, i.e., uniformly between [0, 1], say, or as standard Gaussian variables. Of these the [0,1] uniformly distributed random numbers are key, as the others are obtained using them. Because they can only be generated algorithmically, not "randomly", they are called *pseudo-random* numbers. Sequences of such numbers invariably betray their falsity by repeating their patterns after some cycle length. The generation of pseudo-random numbers – because they would need to satisfy the requirements of their specified probability distributions and mutual independence, etc – is a sufficiently important problem that Knuth devoted to it 200 pages in his encyclopaedic *The Art of Computer Programming*. Improvements in pseudo-random generation have gone on for decades, as it is believed there is no such thing as a *perfect* pseudo-random generator. Yet, we have found one — the Liouville series!

We have seen how the λ's of the Liouville series, while being dependent upon previous λ's of their prime factors, quickly become independent of their immediate predecessors so that the last $n/2$ of the natural sequence of any n λ's are independent of each other, causing an exponential growth of independence further along the series. Even the early dependencies, for small n, are not detected by the χ^2 test, as the numbers involved are too few. So the Liouville series is effectively an infinite *random* sequence of $+1$'s and -1's.

Consider the first few: +1, -1, -1, +1, -1, +1, -1, -1, +1, +1, -1, -1, -1, +1, +1, +1, -1, -1, -1, -1, +1, +1, -1, +1, +1... Let us now convert this to a binary number by interpreting the values to be the digits after a decimal point, with -1's being made 0's, as in: 0.1001010011000111000011011... which gives us an infinite [0,1] uniformly distributed binary number — the first random number generated by the natural number system![41] Further, we can chop up the Liouville series into blocks of M-length and obtain an infinite supply of M-precision [0,1] uniformly distributed binary random numbers[42].

[40]being, seemingly, coin-tosses but actually mostly determined by a structure of dependencies

[41]It possibly deserves a bronze plaque, or something.

[42]which can, of course, be directly converted to standard [0,1] random decimal numbers.

By starting the M-blocks at differently chosen natural numbers P, we can get different random number sequences.[43] This creates a class of *natural random number generators* which give uniformly distributed and never-repeating sequences of *perfect* random numbers of any chosen precision.

I could go on.[44] But, not to further taint the purest of pure mathematics of the Riemann Hypothesis by these atrocious practicalities – as Hardy would have protested – let me now stop.

*

I have now come to the part that I have dreaded writing, although it first gave me the motivation to attempt this essay. It is about the treatment that Kumar's proposed proof has received in the mathematics community.

I can understand how the first sentiment of any mathematician who has some acquaintance with the RH on seeing Kumar's proposed proof would be disbelief. How could this problem that has defeated the greatest mathematical minds of the 20^{th} Century be solved by an *amateur*, and that too in such a simple and straight-forward manner, using no mathematics less than a century old, bypassing without mention all the work, after Littlewood and Hardy's first excursions, on the Riemann Hypothesis – and the Generalised Riemann Hypothesis – by several generations of professional mathematicians? It would, perhaps, constitute the greatest subversion of conventional expectations in the history of modern mathematics. Nevertheless, I would say that professional integrity, if not a more fundamental love for truth and beauty, demands that this work, accessible to every mathematician, should be fairly and openly evaluated.

That has not happened. The proof was completed (except for the LIL part) very rapidly in a few weeks around August-September 2016, and sent to the *Proceedings of the Royal Society – Series A* shortly thereafter, which wrote back after an unexplained delay of more than 8 months (July, 2017) that they would not formally review the paper but helpfully guided Kumar to a website that gave advise to mathematicians who thought they had solved a great problem! They also suggested that the paper be put up

[43]However these sequences need not be unique and independent from each other. Very often, many random number sequences are simultaneously generated and it is important that they not be correlated with each other. Usually they would be of the same precision M (= 32, 64, 128,..bits, say), but would have different starting P's. Even so, the i_1^{th} number of one sequence may fall on precisely the same segment of the Liouville sequence as the i_2^{th} number of another, after which they would become endlessly the same. Such synchronicity, and other unintended correlations, can be avoided by skipping $s + q$ bits of the Liouville sequence after the i^{th} number segment before determining the $(i + 1)^{th}$ number, in each random number sequence, where s is the preassigned unique serial number of the random number sequence (of the many being generated) and q is the (random) binary number indicated by the first Q bits in the Liouville sequence after the i^{th} number segment (e.g., for a prescribed Q=4 and corresponding bits, say, 1010 $\Rightarrow q$=10).

[44]For example, the Liouville infinite binary sequence could be the public key for an encryption algorithm where the private key indicates the segments thereof which stand-in for the keyboard set of alphanumeric characters, and so on.

on a website so that other mathematicians could read and review it. Kumar had already put up the draft on ArXiv.org (September 2016), but then put it on Researchgate along with later updates (the final one in May 2018).

Kumar then send it to *IHES Paris France: Publications Mathmatiques de l'IHS* (Oct 2017), *The National Academy of Sciences, USA* (Jan 2018), *International Mathematics Research Notices* (Feb 2018), and *Ramanujan Journal* (Aug, 2019). In all cases he received a straight-forward refusal to review the paper, or very cursory review, no more than a paragraph long, that dismissed the paper. Appeals for reconsideration were either refused or just ignored. The reviews invariably gave the impression that the reviewer had not read beyond a few pages, merely looking for a quick reason to reject the paper.

More than one editor suggested that Kumar ask a prominent number theorist to carefully read and recommend the paper for formal review. Kumar took up this suggestion. Knowing no number theorists, or any eminent professional mathematicians, he sent the paper with a cover letter to more than 30 mathematicians and theoretical physicists who had published papers on the RH. Just two or three even replied to his letter, and none agreed to review it. Not a single number theorist Kumar corresponded with, then or subsequently, has pointed out a possible fatal error in the proof, or even acknowledged reading the full paper. Kumar is stuck in the classic *Catch-22* — the journals would not formally review his paper till a number theorist had recommended it, and no number theorist would even admit to reading the paper! In the meanwhile, the paper on the Researchgate website, till the date of the this writing, has had several *thousand* downloads — many traceable to prominent academic institutions across the world — but not a single public comment by any self-identified academic mathematician.

Contrast this to the way the world mathematics elite treats one of its own. Within living memory, a longstanding great problem in number theory to be solved was Fermat's Last Theorem. Andrew Wiles, then a professor at Princeton University, proposed a proof in 1993. It was immediately closely examined by other academic mathematicians, and a flaw was found within a few months. Wiles, helped by another mathematician, took a full year to discover a fix for the proof which was finally published in a dedicated issue of the *Annals of Mathematics* in 1995, and Wiles became a star in the constellation of mathematics genius. Wiles proof was 200 pages of very advanced mathematics. Kumar's proposed proof is comprehensible by an undergraduate, and yet, five years on, he still awaits a sincere *review* of his paper by academic mathematicians!

Another great open problem in Pure Mathematics that was recently solved is the Poincare Conjecture, that was proved by Grigori Perelman, a Russian mathematician, around 2003. Perelman, an unconventional person in many ways, did not publish his proof in academic journals but merely uploaded his papers to ArXiv.org and gave a few talks on it at some universities. His work was taken up and its details completed and explained by other mathematicians. Perelman, perhaps to express his disdain for the mathematics establishment, refused all prizes for his work, including the Fields medal.

Western scientific thought, perhaps the greatest intellectual leap of humanity, spread across the world in the 20^{th} Century, which led to an apparent widening of its talent pool, with names like Ramanujan, Raman and Chandrasekhar entering its pantheon. But this democratisation is still largely illusory even today — for one would be hard-pressed to think of a single Nobel Prize winner or a Fields medalist who was not working at some great western university, let alone not educated at one. If genius is everywhere, it certainly is not evident by the geographical distribution of modern scientific achievements.

This has led to a wide prejudice, common in the West but prevalent even in the developing nations[45], that a breakthrough on a great problem would come only from the West by individuals working at a great western university. Kumar, in other words, is an improbability so remote as to be essentially an impossibility. Which is why, it would seem, no number theorist would even bother reading his paper with any seriousness. Just a different address would have ensured that he got all the help and recognition he needed and deserved five years ago, while now he waits to be formally *reviewed* for a breakthrough that would put his name on the front page of every newspaper in the world. He has truly become Ellison's invisible man.

Being Kumar's brother I, of course, take this very personally, more so when sometime ago he transferred all his correspondence on the RH to me, "just in case". Even I, twelve years younger, have started to hear the clock ticking. For him, at 74, it must be like a drumbeat. Let us us hope that it does not come to pass, but if such a dread eventuality were to transpire before his proof is recognised, its poignancy would be of a story from earlier centuries, not the twenty-first.[46] History would be unforgiving to the present-day custodians of mathematics were that to happen. But the loss would not be Kumar's alone. The proof has direct implications for the Generalised Riemann Hypothesis, the problem mentioned as the primary research interest of several of the Kumar's addressees who did not deign to respond to his emails. Who can say what dominos would then fall if those problems are also solved? Mathematics itself awaits acknowledgement of Kumar's proof.

In a larger sense, this has lessons for the way scientific work should be refereed in the 21^{st} Century. The information age, particularly the rise of the internet, has allowed people all over the world direct access to lectures, textbooks and research papers that would have been impossible in earlier times except at some large and well-funded western university. Earlier, people across the world would not even be able to learn *of* the reference material needed to solve a great problem, let alone have it at their finger-tips, as now is possible. In a way, Kumar's proof is a herald of things to come. Incredible though it seems, the Riemann Hypothesis was the low-hanging fruit for the new possibilities opened by the information age. With the large amount of material available on the RH on the internet, it was just a matter of time before someone somewhere would chance

[45]Kumar's paper was sent to at least four recipients of the Bhatnagar Award in Mathematics, India's highest prize in Science and Engineering, with a request for reviews. None responded.

[46]This sounds presumptuous as it assumes the proof is correct, but even if a fatal flaw were to be found, that would be a normal outcome of the scientific process and not, as it now is, its unseemly arrest.

upon the key lying in a darkened doorway, illuminated by a sudden flash of inspiration, while the experts at the great universities were searching for it in the well-lit areas of the main street...

The next great result could come from a small town in Bolivia, or a city in Nigeria. In this new era we will discover genius is everywhere.

*

Afterword

The Riemann Hypothesis is notorious for its power to draw people into obsession. So it was with me. While I understood Kumar's work from its inception, I had no idea of the wider problem and really had no intention of finding out. I presumed, as he did, that he would soon enough be able to know if he was right. Or, even if wrong in the details, he would get help, as Andrew Wiles did, to fix his proof or at least give it a form that would constitute an advancement to the work on the RH. When that did not happen, when Kumar quite literally hit a blank wall with his submissions to journals and correspondence with experts, I, without meaning to, without even wanting to, was drawn in.

I began reading, as much as was necessary, the classic textbooks (and also Derbyshire's popular book [3], which helped greatly) till I understood the steps from Riemann to Littlewood, then on to Kumar's proofs of the Random Walk, ending with the Law of the Iterated Logarithm — and became fully convinced of the rightness of the proof. Stripped to its essentials, it is, as I said, accessible to an undergraduate.[47]

It is also beautiful. The entire passage, of which Kumar's is but a part, from the natural numbers to the zeta function, where the peculiar and seemingly inexplicable behaviour of its zeros was discovered, then on to Littlewood's great insight that converted the inextricable problem in Complex Analysis to a simpler one of series convergence, where again the trick was not to focus on the convergence but on the behaviour of the Liouville integer function which was the coefficient of that series, which, most unexpectedly, brought us to Probability Theory, spans almost the whole of mathematics till Riemann's day and beyond to the probabilists of the early 20^{th} Century, touching every great vista on its way. I am most fortunate to have been among the first to make that journey.

[47] While the motivation to embark on this extended exercise was initially fraternal empathy, the final conclusions were not influenced by that emotion. While tracing the entire path of the RH from its start to Littlewood's analysis and then onto Kumar's steps and then finally to the LIL, which took many months of very concentrated thinking, I was always on the lookout for a fatal flaw that may lie in the details, and I think I came to know those details possibly better than Kumar himself did!

For this and the opportunity to contribute in my own small way, to be as T.S.Eliot wrote,

We, content at the last
If our temporal reversion nourish
(Not too far from the yew-tree)
The life of significant soil.

I shall ever be grateful.

Vinayak Eswaran
Hyderabad, India, 21 February 2021

K. Eswaran's Comments on the Above Review of Professor Vinayak.

There is only one comment rather explanation to make. This explanation is concerned with the Argument given by him in page 10 of his Review:

Since we are comparing the lambda sequence with a sequence of coin tosses say:
c(1), c(2),......,c(n-2), c(n-1),c(n),....
We note as follows the nth coin toss c(n) can have value +1 (H) or -1 (T) and has the properties:
(A) Obviously the value of c(n), the nth coin toss, could be +1 or -1 with equal probability
(B) The value of c(n) does not depend on the previous tosses c(n-1), c(n-2),.. etc.

In other words given any M consecutive values: c(n-M), c(n-M+1), ..,c(n-1), it is impossible to predict the next value c(n). Properties (A) and (B) is sufficient to prove that the distance X(N) travelled in a random walk in N steps, has the square root $C_0 N^{1/2}$ behavior for large N (see Eqs. (21) and (22), Sec 5.2 of my main paper "The Final and Exhaustive Proof…").

Since we wish to show that λ(n) has a similar behavior as c(n) for large n. We therefore have proved "equal Probabilities" and also proved that λ(n) is not predictable if one ONLY knows the M previous values λ(n-1), λ(n-2), ..., λ(n-M), for any finite (fixed) M and large arbitrary n. This was done by proving that the relationship in Eq. (13), in his page 10, is NOT POSSIBLE. **Note: In order that we apply a strict analogy with coin tosses, it was necessary to show that without knowing the explicit value of n, but only the M previous values of λ(n-1), λ(n-2), ..., λ(n-M), it is not possible to predict the next value λ(n), for large n.**
Hence, these results show that the λ - sequence is statistically identical to a sequence of coin tosses (or random walk) for large n and therefore the asymptotic behavior of the summatory Liouville function L(N) for large N, will be identical to the behavior X(N), distance travelled in N steps in a 1-d random walk. Thus proving RH. QED
We have used the fact (or rather the assumption) that mathematical logic, when used with Peano's Axioms in mathematical proofs, should give consistent results. So, if it can be proved that a particular sequence {c(n)} having properties (A) and (B) will satisfy a relation R (say Eq. (22)), then another sequence {λ(n)}, obeying the same properties (A) and (B) must satisfy the same kind of relation R.

KE

---------- Forwarded message ---------
From: **Srinivasan Venkatraman** <vsspster@gmail.com>
Date: Thu, Dec 10, 2020 at 2:12 PM
Subject: Re: Riemann Hypothesis paper review
To: NR RIEMANN <nrriemann@sreenidhi.edu.in>

Dear Madame,
I am hereby sending the Riemann Hypothesis paper review
Re

On Thu, Dec 10, 2020 at 7:02 AM NR RIEMANN <nrriemann@sreenidhi.edu.in> wrote:

Dear Sir,
I am writing this mail to inform you that I have not received the review of Riemann Hypothesis paper.

Thank you sir
Regards
Dr.Suma

The Riemanns.docx
14K

Review of Kumar Eswaran's Papers on the Proof of RH

The Riemann's zeta function is defined as $\zeta(s)=\sum 1/n^s$. Where n is a positive integer and s is a complex number with the series being convergent for Re(s) > 1.

The zeros of this function are at negative even intergers,-2,-4,-6….. There are infinite number of zeros at the line at Re(s) = 1/2. The Riemann's hypotheses claims that these are all the non-trivial zero's of the function. To prove this we investigate the Liouville function λ (n) where: $F(s) = \sum \lambda(n)/n^s$

$$\lambda (n) = (-1)^{\Omega(n)}$$

With Ω(n) being the total number of prime numbers of prime numbers in the factorization of n. Then introduce the function

L(N = $\Sigma^{N}_{n=1}$ λ(n) The partial sum of λ(n).

The independence of λ(n) is shown in the sequence of λ(n), each λ(n) is shown to be independent of the preceding λ's, for large n. The λ (n) is not periodic. The summatory function: L(N)= $\sum_{n=1}^{N} \lambda(n)$ for the large N determines the analyticity of : $F(s) = \sum \lambda(n)/n^s$, using Littlewoods theorem and that the summatory function for the large N mimics the random walk sequence where the sum indicates the distance traveled from the starting point and satisfies the relation

[c(1)+c(2) ……..c(n)] = C. $N^{1/2}$. The random walker behaves, for large n, in such a manner:

1) That each step is of the same size in positive or negative direction and each step occurs in equal probabilities +1 or -1.

2) sequence is not periodic

3) The c's are independent of each other

Also mod L(N) = C $N^{1/2+\varepsilon}$. Thus we see that the lambda behave like coin toss. Thus we can use Khinchine-Kolmogrov law which enable's us to show L(N)/$N^{1/2+\varepsilon}$ = 0 for any episilon greater than 0.

This proves the Riemann's Hypothesis.

Judiciously using the properties of the random walk problem one shows that Riemann's Hypothesis is true.

There is also a numerical proof given.

I complement the author for solving the Riemann's Hypothesis.

Professor V. Srinivasan,

Former Professor University of Hyderabad

December 10th 2020

Summary of Reviews of Papers on RH by K. Eswaran

1 Introduction

28th October 2020

Dear Committee Members,

As you know in February of this year, letters were sent by our Chairman Prof P. Narasimha Reddy, requesting mathematicians and scientists to perform a detailed review of my work on RH. In this report. Nearly 1200 such invitations for review along with copies of my Main paper and links to associated papers were sent. However, only 4 persons have taken up the task of actually doing a detailed review. Though, there was no lack of interest in my work, there have been more than 9000 reads/downloads of my work, probably making it the most popular papers in Number Theory in ResearchGate.

I summarize the reports of the reviewers on the above papers

There were 4 reviewers:
1) Reviewer A: Prof. Ken Roberts, Univ. of Western Ontario, Canada
2) Reviewer B: Prof. SR Valluri. Univ. of Western Ontario,Canada
3) Reviewer C: Prof. Wladislaw Narkiewicz, University of Worclaw, Poland
4) Reviewer D: Prof. German Sierra, Inst. of Th. Physics, Univ. of Madrid, Spain

The summary provided here is an outline of their comments. I have also provided more detailed comments along with their reviews which are attached in three separate pdf files.

2. On Review by A and B

The above two reviewers jointly provided a very detailed review (in more than 20 pages) of the papers and also wrote an Email to the Convener (Dr. A. Suma) summarizing their opinion. Both their email and their report is enclosed along with this write up.

In their email to the convener they have said as follows:

“We found Dr. Eswaran's work quite stimulating of mathematical ideas, and believe that his work should be brought to the attention of a wider scholarly audience, That is, the proof (or selected portions of the methodology) should be published. The Riemann Hypothesis has resisted the efforts of many of the best mathematicians for over 120 years,...”

My Brief comments on their review:

The reviewers have done a detailed study of the concept of towers, of equal probability and of the λ -sequence and they have approved of the methods used. They have also they said that there were parts of the paper that related to the details of the zeta functions which they have not reviewed, but since the main results utilizing these properties has been first done by Littlewood there is no cause for worry. Another aspect they did not comment on is the use of Kechine Kolmogorov Theorem (K-K Theorem) on the Iterative Logarithm which I used to provide the additional (and indisputable) evidence for my proof. This was done by showing that the “width” of the critical line must vanish to zero, hence forcing all the zeros to lie on the critical line. Here again, the KK Theorem is too well known and has been associated with one of the greatest mathematician of the 20th century (Kolmogorov) so there is no cause of worry.

They have also acknowledged that the extensive numerical calculations and statistical study provides evidence of the random-walk behavior of the λ -sequence. Since the behavior of the λ -sequence forms the heart of the proof and they have said: “we have given the first part of the proof, the question of equal probability of the lambda sequence, thorough study”. I feel that it can be said that the RH has been proved.

3. On Review by C

The summary of this reviewer has been extracted from an exhaustive technical discussion with the professor. This email correspondence extended over nearly two months and dealt with the subtitles in the proof. In the beginning the professor was skeptical and I had to explain the procedure and intricacies of the proof in great detail and at the same time convince him of the many objections he raised. *In the end he acknowledged that I do have a proof but the method of proof is rather unusual.* Since the discussion involves nearly 60 pages of mathematics I cannot deal with it in any detail. The entire discussion can be found in thein the attachment which contains the entire verbatim discussion.

I only quote from the last two concluding emails (dated 14 th and 17th April), he stated that my arguments were “heuristic”, though he says “I agree that the similarity of the considered sequence of values of the lambda-function with a random walk gives some reasons to believe in the truth of the conjecture”. I had replied (Apr. 16) to this saying that all I had used is mathematical logic and the well-known properties of numbers, simply put: “It means two mathematical entities which have similar properties will obey similar relationships” denying this, “is equivalent to saying that mathematical logic is an unreliable tool and cannot be trusted if implemented in mathematical proofs”.

His final reply (April 17th 2020) ended with this sentence: “I want also to stress that the word "heuristic" has no negative meaning. A lot of work of really great mathematicians has been performed in a heuristic way. This applies not only to old times (Euler, Laplace, the Bernoulli’s,...) but also to recent times.”

For more details of the review and discussions the attached document may be consulted.

4 On Review by D

Unfortunately this Review is a very incomplete and in fact, if I may say so, an incorrect review.

It all started by a misunderstanding by the Reviewer who thought that (i) I was trying to prove the randomness of primes, a task that I did not even attempt, and (ii) that Equal Probabilities is sufficient to prove RH, because there is no reference by him to the rest of the paper beyond the first few pages. In fact Equal Probability is only one of the requirements for RH to be true there are many more: the proof of unpredictability, the proof of independence and the proof that the λ - sequence is non-cyclic. There is absolutely no reference by him to all the other theorems which were used to prove all these requirements.

I find all this very puzzling because there was no such misunderstanding with the other three Reviewers A, B and C. I had sent him (Reviewer D) a detailed write up not just commenting on his review but explaining the proof and all its intricacies, (it is available as a separate attachment along with his Review) and I suggested that he may please do a relook and a more detailed examination, but he has not responded nor acknowledged receiving my email.

I am submitting this letter with the other documents for your kind perusal. A reading of the comments made by the reviewers would, I hopefully believe, persuade you that the methods and techniques adopted does provide a proof of RH and can stand on its own inspite of the most intense scrutiny/ discussion. (E.g. the mathematical and discussion with Reviewer C, a Polish Number Theorist covered more than 60 pages of print).

I have for your convenience written up a 10 page “Essay on My Proof”, it is available in the file named “Comments_on_Reviewers_A_and_B_Roberts.pdf”

Regards
Dr. Kumar Eswaran
Professor/SNIST
28th October 2020

There are 4 pdf files as attachments along with this document:
1) This document: “Summary of Reviews of Papers on RH by KE.pdf”
2) My Comments and Review by A and B: “Comments_on_Reviewers_A_and_B_Roberts.pdf”
3) “Comments_on_Reviewer_C_Wladyslaw_Narkiewicz.pdf”
4) “Comments_on_Reviewer_D.pdf”

On Ken Roberts and SR Vallur's Review

October 27, 2020

1 CONTENTS OF THIS DOCUMENT

2 My Comments on Review by Ken Roberts and SR Valluri

I thank Professor Ken Roberts and Prof SR Valluri's for their very detailed comments. They have taken upon the onerous task of doing a literature survey and summarizing the work of others as well as reading my paper to great detail.

My Comments:
I will be making my comments more or less in the sequence that they had followed.

1. **Their Summary:** Their summary (pages 1-2) is fine there is nothing for me to comment about.

2. **On BackGround:**
This paragraph briefly states the back ground of the Riemann Hypothesis.

3. **On Equivalent Statements of RH:**
Their writing from pages 3 to 4 is fine.

I took Littlewood's Theorem as the starting point of my proof instead of the Equivalent statement of RH (Eq(6), p. 4). This is because when I had read Borwein's statement attributing the equivalent statement to Landau, I could not find a rigorous proof. I the saw Littlewood's paper and I found that if I choose $F(s) = \zeta(2s)/\zeta(s)$ instead of his $1/\zeta(s)$ I do get a very rigorous proof. The Edward's book also contains Littlewood's method as given by the latter's published paer of 1912 (it's in French). Also, since I wanted to prove RH from first principles (as far as practicable), I used Littlewood's methods and applied it to $F(s)$ so that it is a starting point of my paper. There is no need to validate **statement 1** (their eq (6)), because it is nothing but Littlewood's Theorem which is proved very rigorously by him by using arguments of analytical continuation and has also been re-derived and thoroughly checked by me. In fact a careful study of Littlewood's work leads us to realize that his Theorem is very powerful, **because it says that if (6) is satisfied then RH is True, else it is not.**

4. On Random Walks

Their write up summarizes the issues involved in the study of random walks and its connection with RH.

In answer to the points they have raised in their 1st paragraph (of Sec 3.1) and similar doubts in later on in this section: I only want to ***emphasize*** that we study the statistical properties of $\lambda(n)$, with our attention on Littewood's Theorem which clearly stipulates that we need only to show that for very large values of n $(n \to \infty)$ the $\lambda(n)$ satisfies (i) Equal probabilities and (ii) Independence, if this happens then the statistical distribution of the $\lambda' s$ will be identical to the statistics of a random walk (or coin tosses). The follows from the fact that to prove the $\mid D_N \mid = C.\sqrt{N}$ behaviour of the distance D_N travelled in N steps taken randomly, it is only necessary that the random walk satisfies the above two conditions (i) and (ii). (I am attaching the first 3 pages of the Nobel Laureate Prof Chandrasekar's famous Review paper which contains only these two conditions (i) and (ii) in the derivation of $\mid D_N \mid = C.\sqrt{N}$ behaviour, I have annotated the pages 3 and 4 for the convenience of the reader.)

In another question the reviewers asked if there is any other condition that the $\lambda' s$ need satisfy for the $\sqrt{N}$ relationship to hold? The answer to it is that only (i) and (ii) need be satisfied. If there is some other condition property it is not necessary for the proof.

To justify my statement I have to take recourse to the inherent assumption of the "consistency of mathematical logic in mathematical proofs". I argue as follows: If a particular sequence $\{a(n)\}$satisfies two properties viz Property P and Property Q and if by using the axioms of arithmetic and mathematical Logic we can legitimately deduce that $\{a(n)\}$ satisfies Property R, then any other sequence say,$\{b(n)\}$ which satisfies Property P and Property Q must satisfy Property R. You will agree with me that any other result would call to question the legitimacy of Mathematical Logic in matematical Proofs.

5. **Borwein Integrals and and Random Walks**

The write up about Borwein Integrals was interesting reading.

Much of it was new to me. I am familiar with the use of Sinc function $Sin(x)/x$ in the solution of diffraction problems. But I am not very sure that these Borwein Integrals can directly help in analysing the discrete λ-sequence, because $\lambda(n)$ is an arithmetic function and hence the arithmetic properties of numbers have to be used to discover the behavior of the sequence.

6. Alternative Expression for F(s)

I was aware of the statement that the terms of a series which is not absolutely convergent can be rearranged to get what ever result one wants. But I did not know, till now, that it was Riemann who proved it. Because of the dangers pointed out by this "Rearrangement Theorem", I was very careful in deriving a new representation of $F(s)$ making sure that I start from its defined series representation by choosing choose some s such that $Re(s) >> 1$ ensuring that it was not convergent but absoulutely convergent.

They have also stated that: "We have worked through the details of that example. We are comfortable with this rearrangement of the Dirichlet series representation of F(s) via the Liouville function, because it is being done when everything is absolutely convergent, that is for Re(s) $>$ 1." And they noted that the justification of methods used to obtain the representation can be found in Titchmarsh's book "theory of Functions".

7. Def of Towers $P(m;p;u)$ and Alternative Expression for $F(s)$

They have very nicely explained the concept of Towers and its use in the alternate expression of $F(s)$.

They have taken the trouble to go into great detail explaining each step. I would like to state that by doing so they have travelled almost the exact mental route that first led me to realize that a proof of RH is possible! After writing Eq.3.10) (in my paper) I immediately realised that in each sub-series ('Towers') the $\lambda's$ alternate in sign and therefore the $+1's$and -1s have to be equal. Then by using this new expression of $F(s)$ to analytically continue the function to the regions to the left of the vertical $Re(s) = 1$ line, in the complex s-plane, by using Littlewood's methods, I obtained an equivalent expression for $L(N)$ as depicted by the Fig 1. This figure was crucial to my understanding[1] because a careful examination convinced me that it is the random behavior of the $\lambda(n)'s$ for large n which explains that all the *poles* of $F(s)$ and hence the nontrivial zeros of $\zeta(s)$ lie on the critical line $Re(s) = 1/2$.

8. A Brief Intro On the various proofs of Equal Probability.

Actually, it so turns out that the Prime Number Theorem proves Equal Probability. This proof of course is not mine. I have given two other proofs one is the method of Towers and the other by using the method of constructingn-integers using multiplication of primes and then using induction. So there are actually 3 proofs for Equal Probability. I will write about each of them in turn.

In order to satisfy Littlewoods criteria we need to prove Equal Probabilities of the $\lambda(n)'s$ only for very large n (near or *at infinity*).

Since the $\lambda(n)$ can take only two values viz $+.1$ or -1. And the sample is the whole set of positive integers Z^+. And since we are interested only in N tending

[1] As an after thought, I feel all the things about towers and even the Figure could have be avoided. One can greatly reduce the length of the proof (and the paper) if one just sticks to (1) the def of F(s), and its use in (2) Littlewoods Theorem and (3) prove equal probability and (4) prove Independence to show its statistical similarity to a random walk (coin tosses) and finally (5) invoke the Khinchine- Kolmogorov iterative logarith to filanny prove RH. This procedure would have given a crisp and sharp proof of RH (much like what Gauss would have perhaps done) but would have camouflaged all the insights which led to the proof.

to infinity (or at infinity).[2] There is no need to use complicated definitions of probability and probability spaces. Our definition of probability is simply to answer the question: if we choose an arbitrary integer n (where n can be any number upto infinity),what are the chances that $\lambda(n)$ will be $+1$ or equal to -1.

The equal probabilities theorem can be proved in many ways. (a) By using the Prime Number Theorem[3], (b) by the method of Towers and (c) My the method of Induction,

In the main paper is by forming Towers (every number will be in only one unique tower) and then showing that for each number n with $\lambda(n) = +1$, there is an unique number m (it's twin) with $\lambda(m) = +1$.This would imply that Equal Probabilities is true. If not, suppose that $Prob(\lambda = 1) = 0.75$, this means that there are an insufficiency of integers with their $\lambda's = -1$. This is impossible because every number has "twin" number whose λ-value is it's opposite.

So we no go back to your comments.

8(i) On Equal Probability via PNT (your para 4.1, p 12)

It has been shown that the Prime Number Theorem when interpreted in terms of $\lambda's$ means: $\frac{\lambda(1)+\lambda(2)+\lambda(3)+...+\lambda(n)}{n} \to 0$ as $(n \to \infty)$ (See Borwein etal in footnote[4]). Now since we also know that zeta function has no zero on the line $Re(s) = 1$. We also know from Littlewood's theorem that $L(N) < c.N^{a+\epsilon}$, now, since we know there is no zero at $Re(s) = 1$ we are permitted to put $a = 1$ and $\epsilon = 0$, which leads to $\frac{L(N)}{N} = c$, $(N \to \infty)$, since c is finite (else there will a pole of $F(s)$ at $Re(s) = 1$, meaning a zero of $\zeta(s)$ at $Re(s) = 1$). Therefore, Equal Probability Theorem follows.

The above should clear your doubts that you expressed in paras 3 and 4. The answer is that the PNT does prove Equal Probabilities.

We now move to the proof of Equal Probabilities by Towers.

8(ii) Equiprobability via Towers Argument (para 4.3 p 21)

The concept of towers has been explained in the Main Paper and its appearence in the alternative expression for $F(s)$. The Equal probability follows from the fact that each integer is in a unique tower whose members are in an

[2] It will be very useful to give a meaning to the word *at infinity*. What we do is: to the number system of positive integers $Z^+ \equiv \{1, 2, 3,4,.......\}$ we add the number $\infty$and call this augmented set Z^∞. (Just like in geometry, to the ordinary 2-D Eucledian space A^2we add the point at infinity to get the projective space P^2.) By this definition conceptualization becomes much simpler. (To the purist we can arbitrarily define $\lambda(\infty) = -1$.)

[3] Frankly, at the time of first writing in September 2016, K.Eswaran: The Dirichlet Series for the Liouville Function and the Riemann Hypothesis. I was not familiar with the fact that the PrimeNumber Theorem (PNT), can lead to the proof of Equal Probability". I had seen a paragraph in Borwein's book (see Ref[3] in my Main Paper; also see slide 32 in Peter Borwein's Lecture on RH) stating that Landau had proved that the statement $\frac{\lambda(1)+\lambda(2)+\lambda(3)+...+\lambda(n)}{n} \to 0$ as $(n \to \infty)$ is equivalent to the PNT and this leads to the Equal Probabilities. However, at that time (2016) I could not find a rigorous proof of Equal Probabilities and that is why I took it upon myself to prove it from first principles by the methods of "Towers".

[4] Peter Borwein, Stephen Choi, Brendan Rooney, and Andrea Weirathmueller, 2008, The Riemann Hypothesis: A Resource for the *Afficionado and Virtuso alike*, Canadian Mathematical Society, 2008

ever increasing sequence of integers with alternating λ-values and hence equal number of $+1's$ and $-1's$.

In their comments the Reviewrs A and B gave an accurate description of the building of Towers and its deployment to prove Equal Probability.

8(iii) **Equiprobability (My 2nd Proof)**

I wanted to give a 3rd proof of Equal Probability in such a manner that one need not take recourse to mappings or pairings, but only use induction. This is because many purists consider mathematical induction (especially when used in countable sets) as a more fundamental proof. (I agree that this is a matter of opinion).

I only use the intutive fact that for every odd number there is a unique even number (its successor) and that in the sequence of natural numbers, odd and even alternate therfore the number of odd numbers is equal to the number of even numbers.

We start our demonstration by building up the set of integers

$\{Z^{\infty}\} = \{1, 2, 3, 4, 5, ...\}$ not by addition (as is normally done), but by multiplication of powers of primes. Consider[5] $\{2\} = \{1, 2^1, 2^2, 2^3, 2^4, ...\}$.We see that the $\lambda's$ of each integer in the set alternate i.e equal nos. of $+1$'s and -1's . Hence, $3.\{2\} = \{3.1, 3.2^1, 3.2^2, 3.2^3, 3.2^4, ...\}$ again equal nos. of $+1$'s and -1', we can again go on $3^2.\{2\} = \{3^2.1, 3^2.2^1, 3^2.2^2, 3^2.2^3, 3^2 2^4, ...\}$, which again has equal nos. of $+1$'s and -1'. By continuing and then defining $\{3\} = \{1, 3, 3^2, 3^3, 3^4, ...\}$. We get all the set of integers factorizable by 2 or 3 or both viz. $\{3\}.\{2\} = \{2\}.\{3\}$will have equal nos. of $+1$'s and -1' in their $\lambda's$. The same situation holds for: $\{2\}.\{3\}.\{5\}$, which consists of all integers factorizable by 2 or 3 or 5 or by any combination of their products. We can obtain all integers by writing $\{Z^{\infty}\} = \{2\}.\{3\}.\{5\}.\{7\}.\{11\}$...which will similarly have equal nos of $+1$'s and -1's in their $\lambda's$. This proof employs induction: eg. if $\{2\}.\{3\}.\{5\}$ has equal $+1$'s and -1's in their $\lambda's$, then $\{2\}.\{3\}.\{5\}.\{7\}$ has equal $+1$'s and -1's in their $\lambda's$. And Equal Probabilities must follow (anything else leads to a contradiction). QED

It may be asked: Why do I need so many proofs? My answer is that since this is a proof of RH, my earnest desire is to put all doubts and objections beyond the pale.

9. The Rest of the proof of RH

I have written down a brief essay on RH below which will, I am sure help in explaining the rest of the proof. This has been done for the benefit of the Lay person who wishes to have an overview of the proof.

The following Note (next page) is a stand alone 'Essay ' on the crucial theorems and steps that lead to the proof, I have included it so that any person could help you to undersatnd the rest of the proof.

————————x————————x——————x———————x——————-

[5] Notice that in the exponents of $\{2\}$the number of even and odd exponents are equal and also that their λ- values alternate as $+1$ and -1.

3 Essay on my Proof of RH

You may have noticed that I make no attempt to prove the "randomness of primes" (a conjecture which is very difficult to prove and, luckily for me, is not required for the proof of RH). What I use in my proof is, the randomness of the sequence of $\lambda(n)$ (which appears as the n^{th} term in the Liouville series), in the sense that the λ-sequence (series) resembles a random walk, in that (a) $\lambda(n) = -1\, or +1$ are equally probable and (b) $\lambda(n)$ is not dependent on previous values of the λ's upto any *finite distance*, as the length $N$of the sequence tends to ∞. This statistical resemblence[6] to a random walk for large N, persists even though some individual members of the λ's are related by the deterministic relation $\lambda(mn) = \lambda(m).\lambda(n)$.

These considerations, I then show is enough to prove RH, using the Khinchin and Kolmogorov's Law of the Iterated Logarithm. Both (a) and (b) can be proved in more than one way and I have done so in my paper. Actually, as I have shown, the fact that λ's are deterministic and obey the deterministic relation $\lambda(m.n) = \lambda(m).\lambda(n)$, and therefore not random in the classical sense, does not matter[7] at all for large N. If (a) and (b) can be proved then RH is proved.[8]

There are three things which are very important to the proof. (1) Littlewoods theorem (2) Equal probabilties and (3) Independence. Since (1) and (2) have been already dealt with in the preceding, I will not dwell much on them except to say that I had given two proofs and that at the time of writing that PNT could also be used to povide a rigorous proof of Equal probabilities. It was only later, I became aware that if one uses the Prime Number Theorem *and the fact that $\zeta(s)$ has no zero on the vertical line $Re(s) = 1$ and also Littlewoods theorem (stated in my paper) it is possible to prove Equal Probabilities* see slide 238...in Ref [6] (below)Vinayak's lecture[9]. I have also justified the PNT proof in para 8(i) preceding. Since we seem to have accepted that (a) Equal Probabilities is proved, what remains to prove is (b) that, the $\lambda(n)$ is independent of previous $\lambda's$ (within a finite distance) in a long sequence of length $N \to \infty$. Therefore they would closely resemble the statistical behavior of a ramdom walk (coin

[6]A careful study of Prof Chandrasekar's paper, will make one realize, that to determine the characteristics of a random walk all that Chandrasekar used are two assumptions (i) Equal probabilities and (ii) Independence in a sequence of coin tosses. He requires no other assumption, Hence, since (i) and (ii) are the same as (a) and (b), if these can be proved for large n , this time for the λ- sequence, the required result will follow.

[7]For example, given $\lambda(n)$ for some n, the formula $\lambda(m.n) = \lambda(m).\lambda(n)$,can determine the next predictable value $\lambda(2.n) = \lambda(2).\lambda(n) = -\lambda(n)$, but since we consider very long sequences (length $N \to \infty$) for a large n, say $n = 10^{100}$, the integer $2n$ will be at a distance of 10^{100} from n making such a prediction statistically insignificant! This situation is even more so for the integers in a tower. For example, if there is a large integer j which is in the tower $P(m; p; u)$ and can be written as say $j = m.p^k.u$ then the very next integer (in the tower) above j is the integer $m.p^{k+1}.u$ which is even more far away because generally $p >> 2$.

[8]There are a lot of redundancies in my first paper which, I now think, could have been omitted altogether.

[9]Vinayak Eswaran (my brother), had taken the trouble to write out 7 lectures on my proof of RH, see Ref [6]. The lectures are well worth reading because of their clarity, lack of jargon and assumes only the pre-reqisite of an under-graduate level of mathematical foundation.

toss).

The independence of $\lambda(n)$, for large arguments n,i.e. condition (b), is shown in two different ways. In the first (see Sec 11.3 Appendix IV of Ref.[1]) while it is acknowledged that the $\lambda' s$ are linked through the multiplicative relationship $\lambda(mn) = \lambda(m).\lambda(n)$ etc., it is shown that the numbers linked in this manner move increasingly further from each other so that this distance tends to infinity as $n \to \infty$. Thus in the sequence $\{\lambda(n)\}$ each $\lambda(n)$ is independent of preceding $\lambda' s$ which lie within a finite distance from it as $n \to \infty$.

In the second proof of Independence (see para 2(a), p 21, in Ref[1]), it is shown that any functional relationship which binds $\lambda(n)$ to its previous $\lambda' s$ (which lie within a distance L, arbitrary but fixed) and which is valid for all n,would make the sequence $\lambda(n)$ periodic after some large $n > n_0$. It is then proved that the sequence $\lambda(n)$ cannot have such a cyclic behavior, because this would imply, from Littlewood's theorem, that there are no zeros of the zeta function within the critical strip - which is not true.

With the properties (a) and (b) proved, a direct application of the Khinchine-Kolmogorov Law of the Iterated Logarithm shows that the RH is TRUE and that the width of the critical line is zero!

In the next section onwards I outline the main steps of the proof, which will be a useful to read the paper. (I also strongly recommend Vinayak's Lectures Ref [6] in the Reference Section (below), the lectures provide a detailed, and sometimes an alternative, version of my proof).

In the last I have added an experimental Verification which is not a part of the proof. But the extensive calculations verify the properties of the lambda's over very large sequences and hence lend support and credence to our methods.

4 Gist of The Proof (See Ref. 3 for Details)

The proof proceeds in essentially 4 basic steps.

STEP 1: We choose[10] an analytic function, $F(s) = \zeta(2s)/\zeta(2s)$, whose poles exactly correspond to the non-trivial zeros of the zeta function $\zeta(s)$. $F(s)$ is analytic in the region $Re(s) > 1$and is given by:

$$F(s) = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} \tag{1}$$

STEP 2: An analytical continuation of $F(s)$ to the left of the vertical $Re(s) = 1$, using Littlewood's Theorem determines that the asymptotic behavior of the

[10] It may be mentioned here that for his study Littlewood had chosen the function $1/\zeta(s)$ which had lead to the μ-sequence. A difficulty with using this Mobius function is that $\mu(n)$ can take values$-1, 0, or + 1$, whereas $\lambda(n)$ takes values of $-1\, or + 1$, like a coin toss. Because of this it is easier to compare the λ-sequence with coin tosses rather than the μ-sequence. And because of this very reason I chose to study the function $F(s) = \zeta(2s)/\zeta(s)$, which leads to the λ-sequence.

summatory function $L(N)$:

$$L(N) = \sum_{n=1}^{N} \lambda(n) \tag{2}$$

as $N \to \infty$ plays a crucial role in determining the analyticity of $F(s)$ and the position of the poles of F(s) and thereby the zeros of $\zeta(s)$ in the critical region 0< Re(s) <1. Littlewood's theorem states that the asymptotic behaviour of $L(N)$ for large N, determines the analyticity of F(s), and if the behaviour is such that

$$\mid L(N) \mid \equiv \mid \sum_{n=1}^{N} \lambda(n) \ \mid < C\, N^{a+\epsilon} \qquad (for\, large\, N\,) \tag{3}$$

(where $(1/2 \le a < 1\,)$, and ϵ is a small positive number), F(s) will be analytic in the region $(a < Re(s)$. This is a very crucial result as far as RH is concerned because if one can determine that actually $a = 1/2$ in (3) then the Riemann Hypothesis is proved.

STEP 3: In this step we logically put forth the argument: *that the very necessity that (3) must be satisfied for the Riemann Hypothesis to be true, imposes very severe resrtictions on the behaviour of the sequence of the Liouville functions:* $\{\lambda(1), \lambda(2), \lambda(3), ...\}$. These restrictions (conditions) are given later in this section.

The $\lambda(n)$ is defined as: $\lambda(1) = 1$and for $n > 1$: $\lambda(n) = (-1)^{\Omega(n)}$ and is determined by factorizing n and finding $\Omega(n)$, the number of prime factors of n (multiplicities included). We already know $\lambda(n)$ is fully determined by factorizing n and is an arithmetic function namely: $\lambda(m.n) = \lambda(m).\lambda(n)$, for all integers m, n.

Now for RH to be true one must have $a = 1/2$ in Eq.(3), the first N terms (N large) of the λ sequence must therefore sum up as:

$$\mid \lambda(1) + \lambda(2) + \lambda(3) + ... + \lambda(N) \mid \simeq \ C\,.\,N^{1/2} \tag{4}$$

The above equation brings to mind a similar relationship satisfied by another sequence $c(n) = \pm 1$, which corresponds to the n^{th} step of a One-dimensional random walk! (This $c(n)$ can be simulated by coin tosses, if we replace Heads by +1 and Tails by -1; so a N-step random walk can be thought as a coin toss experiment where a coin is tossed N times.) It is well known that for such a random-walker's sequence the sum indicates the distance travelled from the starting position in N steps and satisfies the relationship:

$$\mid c(1) + c(2) + c(3) + ... + c(N) \mid \simeq \ C\,.\,N^{1/2} \tag{5}$$

The well known result of Equation (5), (see S.Chandrasekar), is derived by using the assumption that the random walker behaves in such a manner that:

(i) Each step is of the same size but can be either in the positive direction or negative direction i.e the nth step $c(n)$ can be +1 or -1, with Equal Probability.

(ii) The sequence of steps cannot be periodic, that is the pattern of steps cannot form a repetitive pattern (there is no cycle) [11].

(iii) Knowing the n^{th} step the $(n+1)^{th}$ cannot be predicted. That is, knowing c(n), c(n+1) cannot be determined (they are independent).

The above assumptions are enough to derive Eq(6). This has been shown by many researchers (e.g. See S. Chandrasekar, refered in Ref[1])

4.1 The Argument:

Comparing (4) and (5) leads us to deduce some inevitable conclusions:

Eq.(4) must be satisfied by the $\lambda(n)$ sequence *if the Riemann Hypothesis is TRUE*, this is the conclusion that we deduce from Littlewoods Theorem, with the proviso that Eq. (4) needs be satisfied only for large N (this being the condition of Littlewood's theorem). Now, there are many Random walks possible, for instance: 100 random walkers can each of them, take N steps and each of these random walkers will be at anapproximate distance of distance $C\,.\,N^{1/2}$from the starting point. Each of these 100 sequences can be thought of as *100 different instances of a random walk of N steps each.*

If we wish to compare (5) with (4) there are several conceptual issues: (α) The sequence in (4) is a deterministic sequence and (β) we have only one sequence. We get over this latter issue by considering the single sequence as *one instance* of a hypothetical random walk of N steps. And even though the $\lambda(n)$'s are deterministic (an aspect we temporarily ignore) we could investigate *this one instance* and argue (or hypothesize) that when N is large, the following rules could be obeyed:

Properties of the $\lambda-$sequence

(a) Given an arbitrary large n chosen at random, there is Equal probability of $\lambda(n)$being either $+1$ or-1.

(b) The $\lambda-$sequence cannot be periodic, that is the $\lambda(n)$ cannot form a repetitive pattern (no cycle)

(c) Knowing the value of $\lambda(n)$ it is not possible to predict $\lambda(n+1)$. Unpredictability (independence).

Note the rules (a),(b)and (c) are similar to (i),(ii) and (iii) and therefore: *If by using the number theoretical (arithmetical) properties of the integers, the primes and the factorization process, it is somehow possible to prove that the $\lambda(n)$ satisy the rules (a), (b) and (c) then just as (5) is satisfied by every instance of a random walk, Eq(4) will be satisfied for our one particular instance of our $\lambda-$sequence and thus RH will be proved! Taking this as a cue we proceed. NOTE: According to Littlewood's Theorem: It is only necessary that the $\lambda-sequence$ satisy the rules for very large lengths of the sequence and large arguments of λ. It is because of this relaxation provided by Littlewoods theorem that even though the $\lambda-$sequence is deterministic, but its behaviour still very closely approximates*

[11] The requirement that there are no cycles was not necessary for Chandrasekar, but it is necessary for the proof of (iii) i.e. independence.

to the statistical behaviour of a sequence of random walks (or coin tosses) over large N.

Hence the next step is to prove the properties for large values of N i.e. when N tends to infinity. (It will also become clear later that the deterministic nature of the $\lambda(n)'s$, does not significantly disturb the above statistical properties.[12])

However, we are actually now at the crossroads: we have to prove that the $\lambda-$sequence possesses the above properties (a),(b) and (c) for large sequence lengths and large arguments N. Property (a) has already been proven, it is quite possible that by using the artithmetic properties of the $\lambda(n)$ that (b) can be proved (as has been done in the Appendix III of the Main Paper) but it is in the proving of (c) that the real difficulty lies. This is because all mathematical proofs in Arithmetic relies heavily on the Axioms of Peano (P.A), but P.A. does not come to our aid for certain hard problems e.g to prove (or disprove) that the advent of primes are random. Luckily we don't have to decide upon this last surmise! But, we do have to decide upon the problem of resolving what is "independence" or "unpredictability" of a sequence. So we define that a sequence $\{a\} \equiv \{a(1), a(2), ...a(n), ...\}$ as unpredictable, for large values of its arguments, if when given the value of $a(N)$ where N is large and the M previous values where $M << N$ (and M finite), viz.$\{a(N-M), a(N-M+1), ..., a(N-1), a(N)\}$ then it is not possible to predict $a(N+1)$. It can be easily seen that if a sequence $\{a\}$ has this property then since, it is not possible to predict the value of $a(N+1)$ knowing $a(N)$ and its M previous values, we can assert that $a(N)$ and $a(N+1)$ are independent. It will be proved that by this definition the $\lambda-sequence \equiv \{\lambda(1), \lambda(2), ..., \lambda(N), ...\}$the components of $\lambda(N)$ and $\lambda(N+1)$, are independent[13] for large values of N. Using this knowledge we can proceed.

STEP 4: Proof of the Properties of the $\lambda-$sequence. In this step several theorems are proved using the number theoretical (arithmetical) properties of integers, primes and the unique factorization of integers to establish the properties (a), (b) and (c) of the $\lambda-$sequence as listed in the previous paragraphs. These proofs are fairly straight forward and are from first principles:

We have already seen that Property (a) On Equal Probabilities, is proved

[12]See Foot note 1, on page 1.

[13]*I define that a sequence* $\{\lambda(n)\}$ *is not Independent*: *If* there is some n_0 and some M (both finite) such that for every $k > n_0$ one can predict $\lambda(k)$ given its M previous values (Note n_0 and M can be very large but must be finite. Also the requirements of Littlewood's theorem are such that these properties need hold only for large n (or k) tending to ∞). That is, there exists some function $f(x_1, x_2, x_3, ..., x_M)$ such that one can write $\lambda(k) = f(\lambda(k-1), \lambda(k-2), \lambda(k-3), ...\lambda(k-M))$. If such a function indeed exists, then one can use it to obtain $\lambda(k)$ and recursively calculate $\lambda(k+1), \lambda(k+2)$,..., for all higher values of n in $\lambda(n)$, this will make the $\{\lambda(n)\}$ predictable. I follow Kurt Godel (See K.Godel: (1931) "On formally undecidable propositions of Principia mathematica and completeness and consistency", pp 592-617, See p. 601 to 602, In Jean van Heijenoort's book "From Frege to Godel", Harvard Univ Press (1967)), by using the concept of recursive functions in our definition of independence. Kurt Godel had said that anything that can be computed (in our case predicted) can be represented by recursive functions. Once the reasonableness of these definitions are accepted then it is not difficult to prove, by using the fact that integers are factorizable uniquely and the arithmetic properties of the $\lambda(n)$ that the $\lambda-$sequence satisfies Equal Probability and Independence.

in Theorems 2 and 3 in Section 5.2, in the Main Paper Ref [1] and that the concept of "towers" is used in the proofs. An alternative proof[14] by constuction of all prime products and induction is also given in Ref [2]. A third proof, which follows from Littlewood's theorem but assumes the fact that there is no zero with Re(s)=1, (proved in the Prime Number Theorem) can also be derived as has been discussed in the foregoing and also by you (but is not given in the paper).

Property (b) On no cycles, is proved in Appendix III, Ref [1]

Property(c) On unpredictability (independence) is proved in Appendix IV, Ref[1]. An alternative proof of this also given: See para 5(a), in page 2, of Ref [3].

An alternative arithmetical proof of the asymptotic relation $\mid L(N) \mid \simeq c = C.N^{1/2}$. is given in Appendix V, Ref [1].[15]

We have therefore showed that Eq. (3) is satisfied by the $\lambda-$sequence. We will now get an expression for the "width" of the critical line and show that this width vanishes in the limit of large N.

We have established the fact that the $\lambda-$sequence behaves like a coin tosses (or a random walk) and this *entitles* us to use Khinchine -Kolmogorov's law of the Iterated Logarithm, adapted to the present context, is: Let $\{\lambda_n\}$ be independent, identically distributed random variables with means zero and unit variances. Let $S_N = \lambda_1 + \lambda_2 + ... + \lambda_N$. The limit superior (upper bound) of S_N almost surely (a.s.) satisfies

$Lim\ Sup\ \frac{S_N}{\sqrt{2N\ loglogN}} \ = 1 \quad as\ N \to \infty$

Now, from Theorem 4 we have written that if we consider the $\lambda' s$ as "coin tosses" one can write $\mid L(N) \mid = \mid \lambda(1) + \lambda(2) + ... + \lambda(N) \mid \leq \ C_0\, N^{\frac{1}{2}+d_N}$ (as $N \to \infty$) (since we are interested in only the behaviour for large N we henceforth ignore the constants). Comparing this expression with the one above we see that one can write $N^{\frac{1}{2}+d_N} \sim \sqrt{N\ loglogN}$ thus yielding an expression for $d_N = \frac{logloglogN}{2logN}$. We see that $d_N \to 0$ as $N \to \infty$, this satisfies the equivalent

[14]This alternative proof of equal probabilities is given in Ref [2] and is done by explict construction of integers by products of sets which are powers of a given prime. Each of these sets are seen to be comprised of members of ascending magnitudes and alternating $\lambda-$values, in perfect analogy with the ascending and alternating odd/even sequences of the natural numbers. Thus the member n of each set has the exact probability of 1/2 (as $n \to \infty$) of its $\lambda-$value being +1 or −1 as the natural numbers have of being odd or even. As every natural number can be placed uniquely in one such subset, it is seen that a randomly chosen natural number will have a probabilty of 1/2 of having its $\lambda-$value equal to +1 or −1(as $n \to \infty$).

[15]In a separate arithmetical study Ref [4], it was discovered that for very large N, smaller primes contribute more (than the larger primes) to the calculation of $\lambda(n)$'s which occur in the summatory expression for $L(N)$. Specifically, if one chooses an integer K such that $K << N$ then the primes p which are s.t $p < N/K$, occur much more often in the calculation of each term in $L(N)$ than the large primes q which are s.t. $N/K < q < N$. This situation permits us to deduce, interestingly, that if we allow both $K\, and\, N \to \infty$ in such a manner that the ratio N/K is a fixed number, then we must have: $Pr(\lambda(n) = 1 \mid n < N) = 1/2 - \frac{C_K}{logN}$ and $Pr(\lambda(n) = -1 \mid n < N) = 1/2 + \frac{C_K}{logN}$ where $C_K$is a small fluctuating number which tends to zero as $K \to \infty$; thus once again confirming that the $L(N)$ behaves like a random walk for very large N.

statement of (3) viz. $\frac{L(N)}{N^{\frac{1}{2}+\epsilon}} = 0$, for any chosen $\epsilon > 0$ (Ref. Eq.(20) in Ref.[1]) . Further $d_\infty \equiv (Lim_{N\to\infty} d_N)$ is the half-width of the critical line. Since this is zero, we conclude that all the non- trivial zeros of the zeta function must lie strictly on the critical line.

Thereby, the Riemann Hypothesis is proved.

5 Experimental verification

In the last Appendix VI, of the Main Paper, Ref[1], numerical experiments (using Mathematica) are described and there it is shown that large sequence of lambdas behave like a random walk (or equivalently like coin tosses).

I have calculated consecutive $\lambda(n)'s$ forming a large sequence, denoted by $\Lambda[N_0, M]$, of length M of the form $\Lambda[N_0, M] \equiv \{\lambda(N_0),\ \lambda(N_0+1),\ \lambda(N_0+2), ...,\ \lambda(N_0+M-1)\}$.where $N_0$is some large integer.

It has been shown by actually computation and performing a χ^2 fit using the methods suggested by Donald Knuth,[16] that these very large sequences containing consecutive values of $\lambda(n)'s$ very closely resemble and are in fact "statistically indistinguishable" from a Binomial distribution ("coin tosses) of equal length. I have done very many computations (using Mathematica) and some of them have been presented in the Tables in Appendix VI,sec.3, e.g. Tables 1.3 and 1.4 page 27; also see "End Note" on the last page of this document. These are accurate and actual computations and the numerical results are indisputable[17].

By very many numerical computations I have shown that the sets of consecutive $\lambda's$ denoted as $S_+(N) = \Lambda(N+1, \sqrt{N})$ and $S_-(N) = \Lambda(N-\sqrt{N}+1, \sqrt{N})$, (Na square integer) have the property of being "statistically indistinguishable" from coin tosses.[18]

These sequences called $S_-(N)$ and $S_+(N)$ exist (N being a perfect square) and behave like random sequences (coin tosses) *and the concatination of such sets of $S_-(N)$ and $S_+(N)$ cover all of $\lambda(n)$ for all integers n upto infinity.* This shows that the entire $\{\lambda(n)\}$ sequence is made up of an infinite series of subsequences of type $S_-(N)$ *and* $S_+(N)$ each of which statistically behave like coin tosses! The Tables 1.4 cited above, provide ample proof of this. The purpose of this section is just to demonstrate that the, predictions of the theorems proved in the Main Paper, have been numerically verified extensively. The verifications[19]

[16] Knuth D.,(1968) 'Art of Computer Programming', vol 2, Chap 3. Addison Wesley

[17] I believe my papers provides the ***raison d'etre*** for the existence of this phenomena.

[18] The reason for this was demonstrably argued because each integer n occurring in the argument of $\lambda(n)$ in one of the sets say $S_+(N)$ belongs to a different "Tower".

Notice that if you choose $N = j^2$then the union of the two sets :

$S_+(j^2+1, j) \cup S_-((j+1)^2 - j, j+1)$ is nothing but the sequence

$\{\lambda(j^2+1),\ \lambda(j^2+2),\ \lambda(j^2+3), ...,\ \lambda((j+1)^2)\}$. That is they cover all the $\lambda's$ with arguments between two consecutive perfect squares, j^2 to $(j+1)^2$. Now if you choose $N = (j+1)^2$ you can cover the next region between the perfect squares $((j+1)^2+1)$ to $(j+2)^2$ and therefore you can capture all the regions between two consecutive perfect squares by concatinating such sets all the way up to infinity - basically covering all integers by the union of sets S_- and S_+right up to infinity.

[19] *In fact I have proved (in Appendix VI of the Main Paper) that if you do a χ^2 fit of*

have been done by doing a χ^2 fit of a $\lambda-$sequence with a Binomial distribution (coin tosses). In every case it has been shown that for large N the $\lambda-$sequence is indistinguishable from a random walk (sequence of coin tosses).

These numerical computations and χ^2correlations are very real and are actually present and give very strong indications of "randomness" present in the $\lambda(n)'s$ which were actually computed by the factorization of integers n. *I strongly believe this phenomena has to be explained by the Pure Mathematicians and not brushed aside or put under the carpet or carelessly labeled as mere coincidence!*

I wish to emphasize, that I have not only explained this phenomena but also showed how it connects with the proof of the Riemann Hypothesis.

6 Conclusion

In this write up I have shown that the reason for the RH to be true lies with the fact that the λ- sequence behaves statistically like coin tosses. It was shown that a sequence $c(n)$ of coin tosses or a sequence of a random walk, exhibits the square root behavoir of Eq(4), was deduced by Khinchine-Kolmogorov, Chandrasekar and others, from the assumption of two criteria (i) Equal Probabilities (ii) Independence. By using the properties of arithmetic and from the use of mathematical deductions we could prove many theorems (and many have alternative proofs) to show that the λ- sequence for large values of its arguments also satiisfy (i) and (ii). And therfore they satisfy Littlewood 's condition for RH to be true. We also made extensive numerical computations involving the $\lambda-$sequence; all these support the various theorems we have proved. Since, we have started from first principles and used only the arithmetic properties of numbers and mathematical logic to prove the theorems, in my opinion, this leaves hardly any doubt as to the truth of RH. I believe that I have solved it comprehensively.

K. Eswaran

7 References

[1] The final and Exhaustive proof of the Remann Hypothesis...

[2]A Simple Proof That Even and Odd Numbers of Prime Factors Occur with Equal Probabilities in the Factor-ization of Integers

[3]A Quick Reading Guide to the Proof of the Riemann Hypothesis

[4] The effect of of the non-random-walk behavior of the Liouville Series L(N) by the first finite number of terms.

a sequence of $\lambda's$ of length N with a sequence of coin tosses of equal length (using Knuth's method) then $\sum_{n=1}^{N}\lambda(m+n) = \chi\sqrt{N}$ for N large. If necessary, one may consult the extensive numerical calculations done at the end of the last Appendix involving $\lambda-$sequences as long as 100,000 and argumets of λ as large as 10, 000,000,000!

I enclose below the slides of the Invited Lecture that I delivered at the Government Arts & Science College Kumbakonam on March 1st 2019. (This was followed by another (slightly shorter) Lecture delivered in the Ramnujan Centre of Sastra University on the evening of the same day).

[5] Invited Lecture On the Riemann Hypothesis by K.Eswaran

[6] Vinayak Eswaran: Seven Lectures of Kumar Eswaran's Proof on RH

K. Eswaran/Professor
Sree Nidhi Institute of Science and Technology,
Yamnampet, Ghatkesar, Hyderabad 501301
10th October 2020

8 END NOTES *-These are just end notes which -provide some more information.*

Just for curiosity I tested the behaviour of very long sequences of lambda and compared them with coin tosses by χ^2 fitting.See the Appendix VI of my Main Paper Ref[1], there are many more examples in the form of Tables.

Here we define the summation: SUM $\equiv \sum_{n=K}^{K+L} \lambda(n) = \chi\sqrt{L}$ (see Eq(9), in page 25 of Ref[1]).

Everywhere the χ^2 fits get better and better as the the length of the sequence and the size of the integer increases. Knuth speculated that a value of around 4 or 5 to make the sequence indistinguishable from a sequence of coin tosses or a random-walk. However, Littlewood's criterion is far less strict, it is sufficient that as N tends to infinity: $\sum_{n=K}^{K+N} \lambda(n) = C\sqrt{N}$, where C can be any finite value.

In the tests below the lambda sequence passes the test in every case. I have taken very Long sequences, for Example III, I have considered 100,000 consecutive integers starting from $K = 25 \times 10^{24} + 1$

EXAMPLE I: A sequence of Length L of consecutive lambdas starting from $\lambda(25000001)$ to $\lambda(25005000)$ of 5000 λ values for 5000 consecutive integers,starting from $25,000,001$ ie $L = 5000$. We use Mathematica commands in our computations as shown below.

Plus[LiouvilleLambda[Range[25000001, 25005000]]]

= {1, 1, 1, 1, 1, 1, 1, -1, -1, -1, 1, -1, -1, 1, 1, 1, -1, -1, 1, 1, \ 1, -1, -1, 1, 1, -1, 1, 1, -1, 1, 1, -1, -1, 1, 1, -1, 1, -1, 1, -1, \ -1, -1, 1, -1, -1, -1, -1, 1, -1, -1, 1, -1, 1, -1, 1, -1, 1, 1, 1, \TRUNCATED-- \ 1, -1, 1, 1, 1, 1, 1, 1, -1}

SUM= -42, χ^2= [SUM * SUM/ L] = [42*42/5000}= 0.3528

EXAMPLE II

Starting Integer $K = 25 \times 10^{24} + 1; L = 10,000$

Plus[LiouvilleLambda[Range[25000000000000000000000001, 25000000000000000000010000]]]

SUM= 88, χ^2 = 88*88/10000 = 0.7744

EXAMPLE III:

Starting Integer $K = 25 \times 10^{24} + 1; L = 100,000$

Plus[LiouvilleLambda[Range[25000000000000000000000001, 25000000000000000000100000]]]
SUM= 238, χ^2 = 238*238/ 100000 = 0.56644

EXAMPLE IV:
Starting Integer Z = $10^{30} + 1$; L=1000;
Plus[LiouvilleLambda[Range[Z, Z + 999]]]
SUM= -20, χ^2 = 20*20/1000 =0.4

EXAMPLE VI:
Starting Integer Z = $10^{30} + 1$; L=10,000;
Plus[LiouvilleLambda[Range[Z, Z + 9999]]]
SUM= 54 χ^2 = 54*54/10000 = 0.2916

9 Email to Convenor Dr. A. Suma from Ken Roberts

Ken Roberts <krobe8@gmail.com> to: NR RIEMANN <nrriemann@sreenidhi.edu.in>
cc: Sree Ram Valluri <vallurisr@gmail.com>, "Dr. Kumar Eswaran" <kumar.e@gmail.com>
Oct 21, 2020, 9:15 AM

Dear Dr. A. Suma,

We are pleased to forward a report, with our remarks on Dr. K. Eswaran's proposed proof of the Riemann Hypothesis. Our report is document PPRH-20201021 dated 21-Oct-2020. This report is joint work of Prof. S. R. Valluri and myself.

Our report is incomplete, in that we did not examine all aspects of the proposed proof. There are some aspects of the Riemann zeta function with which we are not sufficiently familiar in order to speak authoritatively, However, we have given the first part of the proof, the question of equal probability of the lambda sequence, thorough study.

We found Dr. Eswaran's work quite stimulating of mathematical ideas, and believe that his work should be brought to the attention of a wider scholarly audience, That is, the proof (or selected portions of the methdology) should be published. The Riemann Hypothesis has resisted the efforts of many of the best mathematicians for over 120 years, and any advance, even a partial one, is to be communicated. We believe that Dr. Eswaran's towers construction, for instance, is quite innovative and useful. As well, we believe that his paper may prompt the exploration by other scholars of some related topics, including pseudo-random-walk sequences and the Borwein integrals seen as random walks. Details of those suggestions are in the attached report. We are not suggesting that Dr. Eswaran must personally investigate those topics, as they are not directly required for the refinement and validation of his proof. Rather, such related topics represent an opportunity for anciliary investigations by the wider mathematical community.

We appreciate the opportunity to review and comment upon Dr. Eswaran's proposed proof. We hope that all aspects of the proposed proof are correct or can be amended to resolve uncertainties. Even if the proposed proof turns out to be deficient in some manner, it does constitute an useful advance on the Riemann Hypothesis problem. As well, though we have not addressed the topic much in our attached report, we found some of Dr. Eswaran's representations of the behaviour of RH-related sequences to be quite insightful and thought-provoking. Those studies also deserve to be shared with a wider audience.

Best wishes, and thank you, Ken Roberts 21-Oct-2020

copies to: Prof. S. R. Valluri and Dr. K. Eswaran attachment: PPRH-20201021 (pdf file) ATTACHMENT CONTAINING THE DETAILED REVIEW IS IN A SEPARATE PDF FILE

The Simplest One-Dimensional Problem: The Problem of Random Walk

BY PROF. S CHANDRASKAR

For your convenience, I have copy pasted only the first three Pages CHAPTER ONE of Prof Chandrasekhar's Paper:

REVIEWS OF
MODERN PHYSICS

VOLUME 15, NUMBER 1 JANUARY, 1943

Stochastic Problems in Physics and Astronomy

S. CHANDRASEKHAR
Yerkes Observatory, The University of Chicago, Williams Bay, Wisconsin

CONTINUED NEXT PAGE

observation at a later time t. We ask: What can we say about the possible values of Φ which we may expect to observe at time t when we already know that Φ had a particular value at $t=0$? It is clear that if the second observation were made after a sufficiently long interval of time, we should not, in general, expect any correlation with the fact that Φ had a particular value at a very much earlier epoch. On the other hand as $t\to 0$ the values which we would expect to observe on the second occasion will be strongly dependent on what we observed on the earlier occasion.

An example considered by Smoluchowski in colloid statistics illustrates the nature of the problem presented in theories of probability after-effects: Suppose we observe by means of an ultramicroscope a small well-defined element of volume of a colloidal solution and count the number of particles in the element at definite intervals of time τ, 2τ, 3τ, etc., and record them consecutively. We shall further suppose that the interval τ between successive observations is not large. Then the number which is observed on any particular occasion will be correlated in a definite manner with what was observed on the immediately preceding occasion. This correlation will depend on a variety of physical factors including the viscosity of the medium: thus it is clear from general considerations that the more viscous the surrounding medium the greater will be the correlation in the numbers counted on successive occasions. We shall discuss this problem following Smoluchowski in some detail in Chapter III but pass on now to the consideration of another example typical of this theory.

We have already indicated that a fundamental problem in stellar dynamics is the specification of the distribution function $W(\boldsymbol{F})$ governing the probability of occurrence of a force $\boldsymbol{F}$ per unit mass acting on a star. Suppose that $\boldsymbol{F}$ has a definite value at a given instant of time. We can ask: How long a time should elapse on the average before the force acting on the star can be expected to have no appreciable correlation with the fact of its having had a particular value at the earlier epoch? In other words, what is the *mean life* of the state of fluctuation characterized by $\boldsymbol{F}$? In a general way it is clear that this mean life will depend on the state of stellar motions in the neighborhood of the star under consideration in contrast to the probability distribution $W(\boldsymbol{F})$ which depends only on the average number of stars per unit volume. The two examples we have cited are typical of the problems which are properly in the province of the theory dealing with probability after-effects.

A physical problem, the complete elucidation of which requires both the types of theories outlined in the preceding paragraphs, is provided by Brownian motion. We shall accordingly consider certain phases of this theory also.

CHAPTER I

THE PROBLEM OF RANDOM FLIGHTS

The problem of random flights which in its most general form we have already formulated in the introduction provides an illustrative example in reference to which we may develop several of the principal methods of the theories we wish to describe. Accordingly, in this chapter, in addition to providing the general solution of the problem, we shall also discuss it from several different points of view.

1. The Simplest One-Dimensional Problem: The Problem of Random Walk

The principal features of the solution of the problem of random flights in its most general form are disclosed and more clearly understood by considering first the following simplest version of the problem in one dimension:

A particle suffers displacements along a straight line in the form of a series of *steps* of equal length, each step being taken, either in the forward, or in backward direction with equal probability $\frac{1}{2}$. After taking N such steps the particle *could* be at any of the points[3]

$$-N,\ -N+1,\ \cdots,\ -1,\ 0,\ +1,\ \cdots,\ N-1 \text{ and } N.$$

We ask: What is the probability $W(m, N)$ that the particle arrives at the point m after suffering N displacements?

We first remark that in accordance with the conditions of the problem each individual step is equally likely to be taken either in the back-

[3] These can be regarded as the coordinates along a straight line if the unit of length be chosen to be equal to the length of a single step.

ward or in the forward direction quite independently of the direction of all the preceding ones. Hence, all possible sequences of steps each taken in a definite direction have the same probability. In other words, the probability of any given sequence of N steps is $(\frac{1}{2})^N$. The required probability $W(m, N)$ is therefore $(\frac{1}{2})^N$ times the number of distinct sequences of steps which will lead to the point m after N steps. But in order to arrive at m among the N steps, *some* $(N+m)/2$ steps should have been taken in the positive direction and the remaining $(N-m)/2$ steps in the negative direction. (Notice that m can be even or odd only according as N is even or odd.) The number of such distinct sequences is clearly

$$N!/[\tfrac{1}{2}(N+m)]![\tfrac{1}{2}(N-m)]!. \qquad (2)$$

Hence

$$W(m, N)=\frac{N!}{[\frac{1}{2}(N+m)]![\frac{1}{2}(N-m)]!}\left(\frac{1}{2}\right)^N. \qquad (3)$$

In terms of the binomial coefficients C_r^n's we can rewrite Eq. (3) in the form

$$W(m, N)=C^N_{(N+m)/2}\left(\frac{1}{2}\right)^N, \qquad (4)$$

in other words we have a *Bernoullian distribution*. Accordingly, the expectation and the mean square deviation of $(N+m)/2$ are (see Appendix I)

$$\left.\begin{aligned}\tfrac{1}{2}\langle N+m\rangle_{Av}&=\tfrac{1}{2}N,\\ \langle[\tfrac{1}{2}(N+m)-\tfrac{1}{2}N]^2\rangle_{Av}&=\tfrac{1}{4}N.\end{aligned}\right\} \qquad (5)$$

Hence,

$$\langle m\rangle_{Av}=0;\quad \langle m^2\rangle_{Av}=N. \qquad (6)$$

The root mean square displacement is therefore $\sqrt{N}$.

We return to formula (3): The case of greatest interest arises when N is large and $m \ll N$. We can then simplify our formula for $W(m, N)$ by using Stirling's formula

$$\log n!=(n+\tfrac{1}{2})\log n-n+\tfrac{1}{2}\log 2\pi+O(n^{-1})\,(n\to\infty). \qquad (7)$$

Accordingly when $N\to\infty$ and $m\ll N$ we have

$$\log W(m, N)\simeq(N+\tfrac{1}{2})\log N-\tfrac{1}{2}(N+m+1)\log\left[\frac{N}{2}\left(1+\frac{m}{N}\right)\right]-\tfrac{1}{2}(N-m+1)\log\left[\frac{N}{2}\left(1-\frac{m}{N}\right)\right]-\tfrac{1}{2}\log 2\pi-N\log 2. \qquad (8)$$

But since $m\ll N$ we can use the series expansion

$$\log\left(1\pm\frac{m}{N}\right)=\pm\frac{m}{N}-\frac{m^2}{2N^2}+O(m^3/N^3). \qquad (9)$$

Equation (8) now becomes

$$\log W(m, N)\simeq(N+\tfrac{1}{2})\log N-\tfrac{1}{2}\log 2\pi-N\log 2-\tfrac{1}{2}(N+m+1)\left(\log N-\log 2+\frac{m}{N}-\frac{m^2}{2N^2}\right)-\tfrac{1}{2}(N-m+1)\left(\log N-\log 2-\frac{m}{N}-\frac{m^2}{2N^2}\right). \qquad (10)$$

Simplifying the right-hand side of this equation we obtain

$$\log W(m, N)\simeq-\tfrac{1}{2}\log N+\log 2-\tfrac{1}{2}\log 2\pi-m^2/2N. \qquad (11)$$

In other words, for large N we have the asymptotic formula

$$W(m, N)=(2/\pi N)^{\frac{1}{2}}\exp(-m^2/2N). \qquad (12)$$

A numerical comparison of the two formulae (3) and (12) is made in Table I for $N=10$. We see that even for $N=10$ the asymptotic formula gives sufficient accuracy.

Table I. The problem of random walk: the distribution $W(m, N)$ for $N=10$.

m	From (3)	From (12)
0	0.24609	0.252
2	0.20508	0.207
4	0.11715	0.113
6	0.04374	0.042
8	0.00977	0.010
10	0.00098	0.002

Now, when N is large it is convenient to introduce instead of m the net displacement x from the starting point as the variable:

$$x=ml \qquad (13)$$

where l is the length of a step. Further, if we consider intervals Δx along the straight line which are large compared with the length of a step we can ask the probability $W(x)\Delta x$ that the particle is likely to be in the interval $x, x+\Delta x$ after N displacements. We clearly have

$$W(x, N)\Delta x = W(m, N)(\Delta x/2l), \quad (14)$$

since m can take only even or odd values depending on whether N is even or odd. Combining Eqs. (12), (13), and (14) we obtain

$$W(x, N) = \frac{1}{(2\pi Nl^2)^{\frac{1}{2}}} \exp(-x^2/2Nl^2). \quad (15)$$

Suppose now that the particle suffers n displacements per unit time. Then the probability $W(x, t)\Delta x$ that the particle will find itself between x and $x+\Delta x$ after a time t is given by

$$W(x, t)\Delta x = \frac{1}{2(\pi Dt)^{\frac{1}{2}}} \exp(-x^2/4Dt)\Delta x, \quad (16)$$

where we have written

$$D = \tfrac{1}{2}nl^2. \quad (17)$$

We shall see in §4 that the solution to the general problem of random flights has precisely this form.

2. Random Walk with Reflecting and Absorbing Barriers

In this section we shall continue the discussion of the problem of random walk in one dimension but with certain restrictions on the motion of the particle introduced by the presence of reflecting or absorbing walls. We shall first consider the influence of a reflecting barrier.

(a) *A Reflecting Barrier at* $m = m_1$

Without loss of generality we can suppose that $m_1 > 0$. Then, the interposition of the reflecting barrier at m_1 has simply the effect that whenever the particle arrives at m_1 it has a probability unity of retracing its step to $m_1 - 1$ when it takes the next step. We now ask the probability $W(m, N; m_1)$ that the particle will arrive at $m(\leqslant m_1)$ after N steps.

For the discussion of this problem it is convenient to trace the course of the particle in an (m, N)-plane as in Fig. 1. In this diagram, the displacement of a particle by a step means that the representative point moves upward by

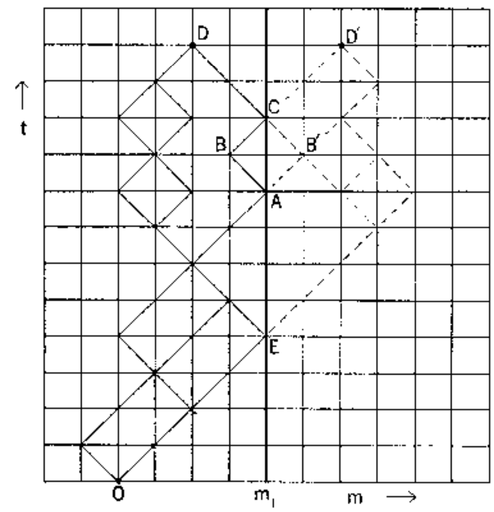

FIG. 1.

one unit while at the same time it suffers a lateral displacement also by one unit either in the positive or in the negative direction.

In the absence of a reflecting wall at $m = m_1$ the probability that the particle arrives at m after N steps is of course given by Eq. (3). But the presence of the reflecting wall requires $W(m, N)$ according to (3) to be modified to take account of the fact that a path reaching m after n reflections must be counted 2^n times since at each reflection it has a probability unity of retracing its step. It is now seen that we can take account of the relevant factors by adding to $W(m, N)$ the probability $W(2m_1 - m, N)$ of arriving at the "*image*" point $(2m_1 - m)$ after N steps (also in the absence of the reflecting wall), i.e.,

$$W(m, N; m_1) = W(m, N) + W(2m_1 - m, N). \quad (18)$$

We can verify the truth of this assertion in the following manner: Consider first a path like OED which has suffered just one reflection at m_1. By reflecting this path about the vertical line through m_1 we obtain a trajectory leading to the image point $(2m_1 - m)$ and conversely, for every trajectory leading to the image point, having crossed the line through m_1 once, there is exactly one which leads to m after a single reflection. Thus, instead of counting twice each trajectory reflected once, we can add a uniquely defined trajectory leading to $(2m_1 - m)$. Consider next a

For your Convenience I have cut and pasted, below, Prof Chandrasekhar's Appendix I.

APPENDIXES

I. THE MEAN AND THE MEAN SQUARE DEVIATION OF A BERNOULLI DISTRIBUTION

Consider the Bernoulli distribution

$$w(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} \quad (p<1;\ x \text{ a positive integer} \leqslant n). \tag{613}$$

An alternative form for $w(x)$ is

$$w(x) = C_x{}^n p^x q^{n-x}, \tag{614}$$

where $C_x{}^n$ denotes the binomial coefficient and

$$q = 1 - p. \tag{615}$$

From Eq. (614) it is apparent that $w(x)$ is the coefficient of u^x in the expansion of $(pu+q)^n$:

$$w(x) = \text{coefficient of } u^x \text{ in } (pu+q)^n. \tag{616}$$

That $\sum w_x = 1$ follows immediately from this remark:

$$\left.\begin{aligned} \sum_{x=1}^{n} w(x) &= \sum_{x=1}^{n} \text{coefficient of } u^x \text{ in } (pu+q)^n, \\ &= [(pu+q)^n]_{u=1} = 1. \end{aligned}\right\} \tag{617}$$

Consider now the mean and the mean square deviation of x. By definition

$$\langle x\rangle_{\text{Av}} = \sum_{x=1}^{n} x w(x) \tag{618}$$

and

$$\delta^2 = \langle (x - \langle x\rangle_{\text{Av}})^2\rangle_{\text{Av}} = \langle x^2\rangle_{\text{Av}} - \langle x\rangle_{\text{Av}}{}^2 = \sum_{x=1}^{n} x^2 w(x) - \langle x\rangle_{\text{Av}}{}^2. \tag{619}$$

CONTINUED ON NEXT PAGE

We have

$$\left.\begin{aligned}\langle x\rangle_{\mathrm{Av}}&=\sum_{x=1}^{n}x\times\{\text{coefficient of }u^x\text{ in }(pu+q)^n\},\\&=\sum_{x=1}^{n}\text{coefficient of }u^x\text{ in }\frac{d}{du}(pu+q)^n,\\&=\left[\frac{d}{du}(pu+q)^n\right]_{u=1}=np(p+q).\end{aligned}\right\}\tag{620}$$

Hence

$$\langle x\rangle_{\mathrm{Av}}=np.\tag{621}$$

Similarly,

$$\left.\begin{aligned}\langle x^2\rangle_{\mathrm{Av}}&=\sum_{x=1}^{n}x^2\times\{\text{coefficient of }u^x\text{ in }(pu+q)^n\},\\&=\sum_{x=1}^{n}\text{coefficient of }u^x\text{ in }\frac{d}{du}\left(u\frac{d}{du}[pu+q]^n\right),\\&=\left\{\frac{d}{du}\left(u\frac{d}{du}[pu+q]^n\right)\right\}_{u=1},\end{aligned}\right\}\tag{622}$$

or,

$$\langle x^2\rangle_{\mathrm{Av}}=np+n(n-1)p^2.\tag{623}$$

Combining Eqs. (619), (621) and (623) we obtain

$$\delta^2=np-np^2=np(1-p)=npq.\tag{624}$$

End of Appendix.

Remarks on a Proposed Proof of the Riemann Hypothesis

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Ken Roberts[1] and S. R. Valluri[2]

October 21, 2020[3]

Summary

This document has our remarks on Dr. Kumar Eswaran's proposed proof of the Riemann Hypothesis. These remarks address the proof's mathematical ideas and methods. We focus on the aspects of KE's proposed proof which are innovative. We do not attempt to place the proof in the context of historical and recent other work on the RH. We accept well known results and adopt the notation and terminology which is used in KE's writings. We find it advisable to elaborate upon some aspects for clarity. We pose a number of questions. Some questions highlight a statement which we have not been able to adequately verify from the cited literature. These are really requests for further detailed information such as location of the statement within a cited document. Other questions call attention to a potential shortcoming in some reasoning. These are requests for further clarity, and are not necessarily critical to the main argument of the proposed proof. Finally, some questions call attention to related topics. These questions are suggestions of possible opportunities for further interesting explorations by KE or others.

KE's writings which were consulted are [KE1, KE2, KE3]. The primary paper is [KE1] and the other two papers provide some background. A convenient brief overview is [KE2] which makes reference to [KE1] for details and justifications. Another helpful reference on the proposed

[1]Email: krobe8@uwo.ca or krobe8@gmail.com, Physics and Astronomy, Western University, London, Canada.

[2]Email: vallurisr@gmail.com or valluri@uwo.ca, Physics and Astronomy, Western University, and Kings University College, London, Canada.

[3]File location: LT4:u4:kwork4:eswaran

proof is a series of seven slide lectures by Dr. Vinayak Eswaran, which are available either at Youtube or as a pdf document [VE1].

KE's proposed proof is quite innovative, and has several interesting ideas. These methods may also be useful for other problems. We are at present undecided on whether the proof is accurate in all respects. There are some aspects which we believe require clarification in order to construct a fully justified proof. We have found the study of KE's proposed proof quite stimulating. It suggests additional topics for investigation, and we appreciate the opportunity to consider KE's work in detail.

At present, we have made a detailed examination of only a part of the proposed proof, the portion which deals with the question of whether the $\lambda(n)$ values (defined below) have equal probability of being +1 or -1. We are forwarding this (incomplete) review at this time, in the hope that it will be useful for Dr. Eswaran and also for others who are examining the proof.

In our opinion, the proposed proof should be published so that it may be examined by a wider community of scholars. Even if the proposed proof turns out to be deficient in some manner, it does constitute an advance on the RH problem, which has been an open question and investigated by numerous experts for over 120 years. Any advance in methodologies is worthwhile as a contribution to the investigation of the Riemann Hypothesis. We believe in particular that Dr. K. Eswaran's towers method is insightful and suggests further opportunities.

In the remainder of this document, we will summarize the proposed proof, and highlight questions which we have. We do not necessarily expect Dr. Eswaran to take responsibility for addressing all questions. We hope that, with wider publication, other scholars will be drawn to this interesting subject as there are many opportunities for work on the details.

1 Background

Peter Borwein and colleagues have prepared an excellent 2008 handbook on the Riemann Hypothesis [PB] which contains a summary of prior work and a selection from the many important papers related to the RH. Another important resource is the 1974 book by H. M. Edwards on the Riemann Zeta Function [HME]. Each book also includes an English translation of B. Riemann's original research report of 1859 in which he first stated his hypothesis.

Riemann considered a function $\zeta(s)$ of a complex variable s which, if $Re(s) > 1$, is given by the Dirichlet series

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + ... = \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{1}$$

Riemann extended the $\zeta(s)$ function by analytic continuation to the whole complex plane, except for the point $s = 1$. The $\zeta(s)$ function has zeros when s is an even negative integer. Those are called the trivial zeros. As well, $\zeta(s)$ has an infinite number of isolated zeros $s = \sigma + \mathbf{i}t$. All known non-trivial zeros of $\zeta(s)$ lie on the line $\sigma = Re(s) = \frac{1}{2}$. That line is called the critical line. The Riemann Hypothesis (RH) is that all the non-trivial zeros of $\zeta(s)$ lie on the critical line.

Two of the early serious investigations of the RH were by T. J. Stieltjes in the 1880s and by J. Hadamard in the 1890s (see [HME], pp. 262-263). Subsequent investigators include many of the great names in number theory and analysis.

2 Equivalent Statements to RH

KE's paper [KE1], following one of the avenues taken in prior investigations, defines a function $F(s)$ by

$$F(s) = \frac{\zeta(2s)}{\zeta(s)}. \tag{2}$$

That function has poles at exactly the non-trivial zeros of $\zeta(s)$ and is convenient to study.

Chapter 5 of the handbook [PB] has several statements which are equivalent to the RH. First we need some definitions.

Define a function $\Omega(n)$ on the natural numbers, letting $\Omega(n)$ equal the number of prime factors dividing n, counting multiplicity. For example, $\Omega(40) = \Omega(2^3 \cdot 5) = 3 + 1 = 4$.

Define the Liouville function $\lambda(n)$ via the formula

$$\lambda(n) = \lambda_n = (-1)^{\Omega(n)}. \tag{3}$$

The Liouville function (LF) is completely multiplicative. The notations $\lambda(n)$ and λ_n will be used interchangeably, as convenient.

The relation between the function $F(s)$ defined above and the Liouville function is that $F(s)$ has the Dirichlet series with terms given by the LF:

$$F(s) = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}. \tag{4}$$

Thus one can approach the Riemann $\zeta(s)$ function via a study of the properties of the Liouville $\lambda(n)$ sequence, and vice versa.

Define the summatory Liouville function $L(N)$ via the formula (see [KE1], equation (1.2b)),

$$L(N) = \lambda(1) + \lambda(2) + ... + \lambda(N) = \sum_{n \le N} \lambda(n). \tag{5}$$

The notations $L(N)$ and L_N will also be used interchangeably.

It is known (see [PB], Equiv 5.2, pg. 46) that the RH is equivalent to **Statement 1:** For any fixed $\epsilon > 0$,

$$\lim \left(\frac{L(N)}{N^{\frac{1}{2}+\epsilon}} \right) = 0 \text{ as } N \to \infty. \tag{6}$$

3 Random Walks

KE's insight is that the Liouville function $\lambda(n)$, can be considered as an unbiased random walk with steps ± 1. That is not a new observation. There is considerable numerical evidence that $\lambda(n)$ looks random. However, KE has given the logic of the idea the serious and detailed consideration which it deserves. The mathematical challenge, as outlined in [KE2], is to develop arguments which make it clear that the properties observed in the expected behaviour of unbiased random walks as a class, are also present in the particular deterministic sequence $\lambda(n)$, and that those properties are sufficient for the $\lambda(n)$ sequence to satisfy the analogue of **Statement 1** above.

Let $c(n)$ denote an infinite sequence of ± 1 values. Let $S(N)$ denote the corresponding summatory function,

$$S(N) = c(1) + c(2) + ... + c(N) = \sum_{n \leq N} c(n). \tag{7}$$

Suppose that the sequence $c(n)$ is an unbiased one-dimensional random walk with steps ± 1. In random walk terminology, the summatory function $S(N)$ is the position of the particular walker c after having taken N steps. That $c(n)$ is unbiased and random means that, for any particular n index, the probability of $c(n)$ being $+1$ is $1/2$. Moreover, the value of $c(n)$ does not depend upon the values of $c(k)$ for the $n-1$ values $k = 1, 2, ..., n-1$. With these assumptions, the probability distribution function of $S(N)$ is a binomial distribution with a mean of zero. The second moment $|S(N)|^2$ of that distribution is N, so the dispersion, the RMS value of $S(N)$, is $\sqrt{N}$. That means the $c(n)$ sequence satisfies
Statement 2: For any fixed $\epsilon > 0$,

$$\lim \left(\frac{S(N)}{N^{\frac{1}{2}+\epsilon}} \right) = 0 \text{ as } N \to \infty. \tag{8}$$

If one can show that $\lambda(n)$, despite being deterministic, behaves "sufficiently like" an unbiased random walk $c(n)$, then it will satisfy **Statement 2**, and hence **Statement 1** will be established, and thence the Riemann Hypothesis will be established. That is the plan of the proof.

3.1 Pseudo-Random Walks and Deterministic Sequences

It is worthwhile to give more thought to the concept of how a deterministic sequence might be "sufficiently like" a random walk. At various places in the text of [KE1] and related papers, the assertion is made that any sequence $c(n)$ (deterministic or not) which shares certain properties with unbiased random walks will necessarily satisfy **Statement 2**. We are not confident of that assertion. It requires further justification, in our opinion. It would be a useful result, and deserves to be explored fully. It presumably derives from a careful working through of the proofs which are customarily given for random walks, with the deductions from randomness of the sequence replaced by appropriate precise conditions on the sequence.

We make the following remarks and ask a variety of questions, in order to stimulate exploration of the topic:
– The essence of the transition from random walks to the properrties of a deterministic sequence, likely consists of moving away from the ideas of "randomness" and "probability" towards "statistics". That is, it is desirable to be able to show that the $\lambda(n)$ and $L(N)$ values possess certain statistical properties, and that any sequence with those statistical properties will have the property required for **Statement 2** to be true of it.
– Can there be sequences $c(n)$ which do not satisfy **Statement 2**? Certainly. But they may not be unbiased (will have nonzero mean) or may contain periodicities or correlations between finite-scope subsequences (hence not be random). Or ... ?? In what other ways might a sequence fail to satisfy **Statement 2**?
– Suppose that we identify certain properties which is it reasonable to expect of a random walk – such as those mentioned above. Imagine

that such a sequence is deterministic, produced by some algorithm. Call such a sequence a "pseudo-random-walk" (of some type, depending upon the properties which have been chosen to designate that type of pseudo-RW). Can one show that ALL pseudo-RW sequences of that type will satisfy **Statement 2**? For instance, is there perhaps a summability-type argument, and an associated Tauberian theorem, which would let one conclude that property of a specific sequence from the behaviour of an ensemble of sequences?

– The properties of a pseudo-random sequence which are asserted to cause it to satisfy **Statement 2** are, for instance (see [KE1], section 5.2, point 2 in the discussion of the $\lambda(n)$ sequence) that the values of the $c(n)$ are ± 1 with equal probability, and that the values are non-cyclic. Are there other properties required? For instance, the absence of bounded-length cycles may not be enough to ensure that for each n value, the value of $c(n)$ does not depend upon the values of $c(k)$ for the $n-1$ values $k = 1, 2, ..., n-1$.

– It should be understood that, in these remarks, we are not necessarily asking whether the specific sequence $c(n) = \lambda(n)$ might satisfy **Statement 2**. Rather, we ask whether any non-cyclic equiprobable sequence of ± 1 must necessarily satisfy **Statement 2**. Is it possible that some particular additional feature of the $\lambda(n)$ sequence is also essential, such as being completely multiplicative, in order for any non-cyclic equiprobable $c(n)$ to satisfy **Statement 2**?

– It might be, in fact, interesting to explore sequences $c(n)$ which are completely multiplicative but satisfy **Statement 2**. There should be a family of such sequences, for example by setting $c(p) = 0$ for some subset of the primes and $c(p) = \pm 1$ for other primes.

– A general study of pseudo-RW sequences may be worthwhile. Like most mathematical topics, someone has probably already investigated this, though perhaps not with the perspective brought to the task via these RH investigations. So one asks, what work has already been done on pseudo-RW sequences?

– For a proof of the RH, this desired result (that ANY pseudo-RW sequence satisfies **Statement 2**) is not strictly necessary. The sequence $\lambda(n)$ has a particular structure, being completely multiplicative, that may suffice. The details of lecture 6 of [VE1] may suggest some ideas of properties of a sequence $c(n)$ that might be relevant. We are more confident of reasoning which is also based upon multiplicative prop-

erties of the sequence, rather than an assertion regarding all possibly deterministic sequences with equal probabilities and independent subsequences.
– Attempting to construct counterexamples may be an clarifying exercise. That is, to take some pseudo-RW sequence (of a certain type, ie satisfying well defined conditions) and, since we are allowing it to be deterministic, modifying it in such a way that it retains the properties characterizing its type, but has other properties rather different from a pure random walk.

That is rather a scattershot list of questions, with some overlap and repetition. We pose these questions partly to indicate some opportunities for interesting explorations which might come from a consideration of the details in KE's proof. However, that list is also an indication of some discomfort on our part. Without a better grasp of at least some of those topics, we are not fully convinced that the reasoning in the proposed proof is complete.

3.2 Borwein Integrals and Random Walks

We also wish to mention another topic, relevant to the proof structure, prompted by the towers construction illustrated in Figure 1 of [KE1].

The Borwein integrals have an interpretation in terms of random walks. See [DJB] for the original paper which defines the integrals which came to be called the Borwein integrals. See the various references listed at [WBI] for background on these integrals, including the relationship of these integrals to random walks.

The Borwein integrals are Fourier cosine transforms of a finite product of scaled sinc functions. The function $\mathrm{sinc}(x)$ equals $\sin(x)/x$ if x is nonzero, and $\mathrm{sinc}(0) = 1$ for continuity. For $a > 0$, define the modified characteristic function χ_a of the interval $[-a, a]$ by $\chi_a(x) = 1$

if $-a < x < a$, $\chi_a(\pm a) = 1/2$, and $\chi_a(x) = 0$ for x outside $[-a, a]$. The Fourier cosine transform (FCT) of χ_a is $a\sqrt{2/\pi}\,\mathrm{sinc}(ax)$. Conversely, the FCT of the latter function is equivalent to χ_a. We can also interpret χ_a as the probability distribution of a random variable which makes one step, of length uniformly chosen between 0 and $a > 0$, equiprobably in either a positive or negative direction.

Suppose that a random walker takes two steps, each step being equiprobably in the plus or minus direction, and of length uniformly between 0 and a, and between 0 and b. The ending point is the convolution of the functions χ_a and χ_b. The FCT of that probability distribution for a two-step walk is an integral of the product of two sinc functions.

If the random walker takes n steps of lengths uniformly chosen between 0 and a_k, equiprobably in plus or minus directions, for $k = 1, ...n$, then the probability distribution of the walker's ending point is an integral of the product of the n sinc functions.

The essential fact about the Borwein integrals is that, if the sum of the maximum step lengths is sufficiently short then the FCT integral, that is the integral of the product of the sinc functions, equals $\pi/2$. However, if the sum of the step lengths is long enough, then the integral of the product of the sinc functions is less than $\pi/2$, by a rational number factor C which can be exactly calculated.

We do not want to take up too much space in these remarks, but give the following example, which led to the paper [DJB]. The paper has much more than just this example, and suggests many opportunities for further exploration.

Here is the example:
For $n = 0, 1, ...$, define $a_n = 1/(2n + 1)$, and $s_n = a_0 + a_1 + ... + a_n$. Define τ_n by

$$\tau_n = \int_0^\infty \left(\prod_{k=0}^{n} \mathrm{sinc}(a_k\, x) \right) dx. \qquad (9)$$

Then $\tau_n = \pi/2$ for $n = 0, 1, 2, 3, 4, 5, 6$ but $\tau_n < \pi/2$ for all larger n. That is, written out explicitly,

$$\tau_0 = \int_0^\infty \Big(\mathrm{sinc}(x)\Big)\, dx = \frac{\pi}{2}, \tag{10}$$

$$\tau_1 = \int_0^\infty \Big(\mathrm{sinc}(x)\,\mathrm{sinc}(x/3)\Big)\, dx = \frac{\pi}{2}, \tag{11}$$

$$\tau_2 = \int_0^\infty \Big(\mathrm{sinc}(x)\,\mathrm{sinc}(x/3)\,\mathrm{sinc}(x/5)\Big)\, dx = \frac{\pi}{2}, \tag{12}$$

... and so on up to $n = 6$ ...

$$\tau_6 = \int_0^\infty \Big(\mathrm{sinc}(x)\,...\mathrm{sinc}(x/13)\Big)\, dx = \frac{\pi}{2}. \tag{13}$$

However, for $n > 6$, we have $\tau_n < \pi/2$, as seen for the case $n = 7$,

$$\tau_7 = \int_0^\infty \Big(\mathrm{sinc}(x)\,...\mathrm{sinc}(x/13)\,\mathrm{sinc}(x/15)\Big)\, dx = C_7\,\frac{\pi}{2} < \frac{\pi}{2}. \tag{14}$$

The factor C_7 in τ_7 is given by

$$C_7 = 1 - \frac{(w-1)^7}{2^6 \cdot 7! \cdot w}, \tag{15}$$

where

$$w = \frac{1}{3} + \frac{1}{5} + ... + \frac{1}{15}. \tag{16}$$

The interpretation of that property in terms of random walks, is this: Because $1/3 + 1/5 + ... + 1/13$ is less than 1, a random walk of 6 steps of lengths up to those maxima will lie within the interval $[-1, 1]$. However, appending an additional step of length up to $1/15$ will produce a total walk length which may lie outside the interval $[-1, 1]$.

The result is quite general. By making the steps shorter, one can obtain more sinc integrals in the product before the pattern breaks down.

Consider the diagram in figure 1 of [KE1]. It shows an interpretation of the summatory Liouville function $L(N)$ in terms of a random walk with equiprobable steps of length ± 1.

Remarks and Questions:
– There is perhaps a relationship between the concept of the summatory Liouville function $L(N)$ as the ending point of a random walk, and the interpretation of Borwein integrals as random walks.
– It is unlikely that the Borwein integrals can be used to produce a counterexample to the Riemann hypothesis. The RH has been explored to such an extent, via zero finding, that a simple example such as a Borwein integral with steps say of $1/p$ where the p values are primes, is probably not the source of a difficulty. Nonetheless, what might be the structure of an attempt to use the Borwein integral effect to construct an RH counterexample?
– Otherwise posed, the question becomes: What is it about the Borwein integrals represntation of the probability distribution of the endpoint of a random walk, and the Liouville function or other representation of a random walk which allows a random walk based upon, say, steps of size $1/p$ to succeed, ie comply with the Riemann Hypothesis? Is it finiteness of the integrand product in the Borwein integrals? What about infinite analogues of the Borwein integrals?
– What happens if we imagine transforming some Liouville function representation via an analogue of the Fourier cosine transform?
– There are related papers, linked via the Borwein integral references, mentioned above, that bear on such questions.
– Whittaker and Watson [WW], section 6.24, exercise 6 on page 122 discuss these integrals . The result is partly due to Störmer.

We emphasize that our interest in the Borwein integrals was stimulated by KE's proposed proof of the RH, and in particular his intriguing figure 1 in [KE1]. It represents an opportunity for related investigations. We are not expecting those investigations to be done as part of the validation of the proposed proof of the RH.

4 Equal Probability of $\lambda(n)$ Values

The first objective in KE's proof is to establish that the $\lambda(n)$ values have an equal probability of being +1 or -1. There is strong numerical evidence for that claim, but what is required is a math-logic validation. Ideally the proof will be stated in statistical terms, not probabilistic terms, in order to avoid ambiguity related to invoking the concept of probability for a deterministic sequence.

The proof of equal probability which is most detailed and also, in our opinion, the most satisfactory of the alternative proofs presented, is the towers proof. That proof uses a partitioning of the natural numbers into subsequences called "towers" (see [KE1], first couple of pages of section 5.2) in a clever rearrangement of the natural numbers to ensure alternation of the $\lambda(n)$ values because successive n values within the rearrangement have alternatively an odd or even number of prime factors.

In a moment we will move on to the argument for equal probability which uses towers. However, we first wish to mention a theorem on rearrangements of conditionally convergent series, which as it happens is due to Riemann.

4.1 Riemann's Theorem on Rearrangment of Series

Riemann's theorem is described in section 28 of Bromwich's book on Infinite Series, page 74 of the second edition or page 68 of the first edition [BIS], and in section 44 of Knopp's book on the Theory of Infinite Series [KTIS], page 318.

Riemann's Series Rearrangement Theorem: Suppose a series of real numbers converges to a finite sum, but it does not converge absolutely. Choose any real value S. Then the terms of the series can be rearranged, so that it converges to the sum S. It is also possible to rearrange the terms of the series so that it diverges to positive or negative infinity, or so that it oscillates between any two finite or infinite limits.

One proof involves dividing the terms of the series into two ordered sets, the negative terms in one set and the positive terms in the other set, with the zero terms placed in either set. Because the series is convergent, the terms in each set tend towards zero. Because the series is not absolutely convergent, the terms in each set add to respectively negative and positive infinity. The result follows via choosing a rearrangement of the terms which makes the partial sums oscillate on either side of the desired limit, or close to desired lim-inf and lim-sup values. The resulting rearrangment does not alter the relative order of the negative terms, or the relative order of the positive terms, but merely changes the manner in which the two ordered subsets are interwoven.

Remarks and Questions:
– Series which are convergent but not absolutely convergent are called conditionally convergent.
– This result indicates that a single conditionally convergent series can be rearranged to achieve any desired behaviour. However, it is not clear that a series whose terms depend upon a parameter, either a continuous or perhaps a discrete parameter, can be rearranged to achieve an approximation to a target function or values. That is closer to the situation with the $\lambda(n)$ series since multiplicity imposes constraints. Alternatively, if considering the $F(s)$ function via its Dirichlet series, one has a parameter s in play which imposes other constraints.
– The availability of the parameter s will be important in the equiprobability proof via towers of [KE1], which will be considered shortly. The proof via towers works because there is a function $F(s)$ whose representation via towers involves a rearrangement of the Dirichlet series. That rearrangement will preserve the sum, for $Re(s) > 1$, because the Dirichlet series of $F(s)$ is absolutely convergent for such s. Hence with a bit of extra reasoning (done in [KE1]) one gets a representation of $F(s)$ with alternating signs of the $\lambda(n)$ numerators. Details to be discussed below.
– Modifying a conditionally convergent series by adding an absolutely convergent series, or omitting a subseries of terms which are absolutely convergent, will not alter the fact that the series is conditionally convergent.

4.2 Definition of Towers $P(m; p; u)$

Now we turn to the towers proof of equal probabilty. For convenience we are writing $P(m; p; u)$ for towers instead of the notation $P_{m;p;u}$ of [KE1]. Also, we will write (m, p, u, e) for the label which is assigned to a natural number $n \geq 2$, rather than the notation (m, p^e, u) of [KE1]. Section 2 of [KE1] defines towers, and Appendixes 1 and 2 provide additional justification of the definitions.

Natural numbers with one or more prime factors of multiplicity greater than 1 will be considered class I. In contrast, numbers with no prime factors of multiplicity 2 or more will be considered class II. Class II numbers are called square-free numbers elsewhere in the literature. We do not know a short word for numbers which are not square-free, so will simply call those non-square-free. Given a number n, we will call the primes which appear in the factorization of n with multiplicity 2 or more the multiprimes of n, and the primes which appear in the factorization of n with multiplicity 1, the uniprimes of n. Class I is the non-square-free numbers, which have one or more multiprimes. Class II is the square-free numbers, which have no multiprimes, only uniprimes.

If $n \geq 2$ is non-square-free (class I), let p denote the largest multiprime of n, and let e denote the multiplicity of p in n. The value of p is used to split the prime factors of n into two subsets, lesser primes and greater primes. Let m denote the product of all lesser prime factors of n (to their appropriate multiplicities if they are multiprimes in n), and let u denote the product of all greater prime factors of n. All the prime factors in u are necessarily uniprimes, since p is the largest multiprime of n. Then the class I number n is represented by $n = mp^e u$. That is equation (2.7) in [KE1]. The label (m, p, u, e) is attached to n.

If n is square-free (class II), then define p as the largest prime factor of n, define m as the product of all lesser prime factors of n, set $u = 1$ and set $e = 1$. All the factors of such an n are uniprimes, so the product

m is also square-free. The label $(m, p, 1, 1)$ is attached to n. The same equation applies, $n = mp^e u$ for a class II number, and in this case $u = 1$ and $e = 1$ so $n = mp$.

Every $n \geq 2$ has thus been assigned a label (m, p, u, e).

The tower $P(m; p; u)$ is defined to be the ordered set of all numbers n with labels (m, p, u, e) for any value of e. Otherwise visualized, each natural number $n \geq 2$ lies in a 4-dimensional space along m, p, u, e axes, and the tower sets $P(m; p; u)$ are sets of points in the 3-dimensional (m, p, u) space obtained by projecting (m, p, u, e) along the e-axis. The elements of the tower are considered to be in ascending order.

Some towers are empty sets. For instance, if m has a prime factor bigger than or equal to p, or if u has a prime factor lesser than or equal to p, then there is no tower $P(m; p; u)$. Or rather, that tower is an empty set.

Furthermore, if $u > 1$, then any n in $P(m; p; u)$ must be a multiprime, because a square-free n would have been placed in a tower whose label has $u = 1$. Also, if m has a multiprime factor, then all numbers in the tower $P(m; p; u)$ must be non-square-free (class I), because they are multiples of m.

The tower $P(m; p; u)$ will be given by either equation (2.8) or equation (2.9) of [KE1], which are respectively

$$\begin{aligned} P(m; p; u) &= \{mp^2u, mp^3u, mp^4u, ...\} \text{ and} && (17) \\ P(m; p; 1) &= \{mp, mp^2, mp^3, ...\}. && (18) \end{aligned}$$

Are these two equations consistent? KE verifies the consistency, and we agree. Any inconsistency would arise if $u = 1$, and concern the tower into which a number $n = mp$ would be placed. That would only happen if m is a product of primes less than p, and each such prime occurs in m with exponent 1. In such a situation, the square-free value $n = mp$ is class II, and appears as the first element of the tower

$$P(m; p; 1) \;=\; \{mp, mp^2, mp^3, ...\}. \qquad (19)$$

Supposing m a product of uniprimes less than p, the other equation for $u = 1$,

$$P(m; p; 1) \;=\; \{mp^2, mp^3, mp^4, ...\}, \qquad (20)$$

would be incorrect. That tower $P(m; p; 1)$ also includes the class II number $n = mp$, so is given by

$$P(m; p; 1) \;=\; \{mp, mp^2, mp^3, ...\}. \qquad (21)$$

Thus, the towers are well defined. The towers which contain a class II number contain only one such number, at the start of their ordered set. Such towers are $P(1; p; 1)$ which is all the powers of a prime p, and $P(m, p, 1)$ which contain a product of m (formed from primes smaller than p, all primes in m being uniprimes), times all the powers of p. Those are the towers which show up in equation (3.10) of [KE1] with their summation index starting at $r = 1$ instead of $k = 2$.

With these definitions every number $n \geq 2$ is in exactly one tower, and all towers are disjoint. Each tower's sequence of increasing numbers has alternating values of $\lambda(n)$, starting with either +1 or -1 depending upon the particular values of $\lambda(m)$ and $\lambda(u)$ and the initial power of p which appears in the tower: that is, $\lambda(p) = -1$ or $\lambda(p^2) = +1$.

4.3 Alternative Expression for $F(s)$

The function $F(s) = \zeta(2s)/\zeta(s)$ has its poles at the non-trivial zeros of the Riemann zeta function, and has as its Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}. \qquad (22)$$

That series, the Dirichlet series of the Liouville function, is absolutely convergent for $Re(s) > 1$. Section 3 of [KE1] derives an alternative expression for $F(s)$. The expression involves a rearrangement of terms in the Dirichlet series. The rearrangement is valid if $Re(s) > 1$ since the Dirichlet series is absolutely convergent for such s.

Using towers, the paper [KE1] presents a representation of the Dirichlet series with the summation rearranged into a sum of subseries. Each subseries is a summation over a tower. Each tower has alternating values of $\lambda(n)$, so the probability of $\lambda(n)$ being say $+1$ in a particular tower is exactly $1/2$.

The towers construction and the rearrangement of the Dirichlet series to an alternative expression are best understood via the example which is shown in equation (3.10) of [KE1]. We have worked through the details of that example. We are comfortable with this rearrangement of the Dirichlet series representation of $F(s)$ via the Liouville function, because it is being done when everything is absolutely convergent, that is for $Re(s) > 1$. Dirichlet series are unique (more precisely stated in Theorem P2 below), so that any Dirichlet series which also represents $F(s)$ for $Re(s) > 1$ must be the same $\lambda(n)$ series. That is the meaning of the statement towards the end of section 3 of [KE1], that the $h(n)$ numerators in the Dirichlet series of equation (3.16) equal the $\lambda(n)$ values, as per equations (3.18a,b,c).

Remark: The uniqueness of a Dirichlet series representation of a function can be found in Titchmarsh's Theory of Functions (see [TTF], section 9.6, pg 309) or in Hardy and Wright's Theory of Numbers (see [HW], section 17.1, page 245).

Theorem P2: If

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = \sum_{n=1}^{\infty} \frac{b_n}{n^s}, \tag{23}$$

for s in some region (open set) of values of s, then $a_n = b_n$ for all values of n.

4.4 Representation of Summatory Liouville Function $L(N)$

Most of section 4 of [KE1] is concerned with the explanation of figure 1 of the paper. We found that very interesting and suggestive. The figure is not part of the proof but it provides some clarity, and suggests ideas for further exploration. Our remarks made earlier, regarding the Borwein integrals interpreted as random walks, were partly prompted by figure 1. KE has a similar insight, as illustrated by his mention of figure 1 being a representation of $L(N)$ by rectangular waves. Exploration of this topic is likely worthwhile in general, not just for its immediate relevance regarding a proof of the Riemann hypothesis.

The last paragraph of section 4 mentions section 6 of the paper [KE1], which is presumably a reference to section 5.2.

The rearrangement of order of the terms in the Dirichlet series for $F(s)$ is a bit unusual. Each tower is an infinite sequence, so the representation is a sequence of sequences. All terms being absolutely convergent for $Re(s) > 1$ is the condition which allows such an unusual rearrangement.

A question which arises, is whether the equal probability result for the $\lambda(n)$ sequence is correct in a statistical sense, on finite segments $[1, N]$ of the natural numbers. For each N, define N^{pos} to be the count of n values satisfying $1 \leq n \leq N$ fpr which $\lambda(n) = +1$. Define $Q(N) = N^{pos}/N$ to be the fraction of n values in $[1, N]$ for which $\lambda(n) = +1$. The statement that $Prob\{\lambda(n) = +1\} = 1/2$, means that, as $N \to \infty$, $Q(N) \to 1/2$. A rearrangement of the sequence, as such, does not necessarily imply that $Q(N)$ tends to a limit. It is conceivable that $Q(N)$ might oscillate between two distinct values (lim-inf and lim-sup) instead of tending to a limit. Either a rigourous proof or an attempt to construct a counterexample seems in order.

If "equal probability" is not defined as in the prior paragraph, then we might be working with a new concept. Traditional random walk reasoning might not be applicable. The towers rearrangement is interesting, and suggests a way of moving forward towards a proof of the RH. However, it may be necessary to consider a new variety of random walk, one which is carried out on an infinite ordered set.

This is also a concern because the rearrangement of the series $F(s)$ represents $F(s)$ as a sum of zero-hugging subsequences (the towers). Each tower is a deterministic walk which never gets more than ± 1 step away from zero.

The solution to this is no doubt to utilize special properties of the $\lambda(n)$ sequence, for instance its relation to the primes, and also that λ is a multiplicative function.

5 The Remainder of the Proof

We have not examined the remainder of the proposed proof in detail. The most important point we have not considered is whether the $\lambda(n)$ values are non-cyclic. As we understand the point, the question is whether there can be a finite-length relationship which allows one to obtain the value of $\lambda(n)$ from knowledge of a bounded number of predecessors – for instance, from $\lambda(n-k), \lambda(n-k+1), \ldots \lambda(n-1)$, where k is a fixed number and n is allowed to become arbitrarily large. We wonder, however, that even if that point is answered in the negative (ie, the $\lambda(n)$ sequence is non-cyclic), whether there can still be other relationships among the λ values which might prevent it from being a random walk in terms of statistical properties.

We have not given the remainder of the proof in [KE1] the detailed consideration that it deserves. We have, however, identified some questions which are worth further investigation. For instance, regarding the Borwein integrals, some queries which were inspired by the paper [KE1].

We hope that all aspects of the proposed proof are correct, or can be amended to resolve uncertainties. The paper has inspired us to some new opportunities. For that reason we believe the methods (towers, eg) of the [KE1] paper should be published so they are made known to a wider community of scholars. We are advancing our remarks and questions in this report with the sincere hope that they will stimulate other constructive discussion and effort towards proving the Riemann Hypothesis and exploring various related topics such as the Borwein integrals and pseudo-random-walks.

We appreciate the opportunity to review and comment upon Dr. Eswaran's proposed proof.

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Each video lecture displays a sequence of slides, without audio. The slides as a pdf file are available at
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[WW] E. T. Whittaker and G. N. Watson, 1927, *A Course of Modern Analysis*, 4th edition, Cambridge University Press, reprinted 1963.

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[end]

WIKIPEDIA

Władysław Narkiewicz

Władysław Narkiewicz (born February 19, 1936) - Polish mathematician, known for his work on algebraic number theory , algebra and history of mathematics. Full professor since 1974.

Władysław Narkiewicz	
	
Country of Action	Poland
Date of birth	February 19, 1936
Professor of mathematical sciences	
Specialty: number theory, algebra, history of mathematics [1]	
Alma mater	University of Wroclaw
Doctorate	1961 University of Wrocław
Habilitation	1967 University of Wrocław
Professorship	1974
Academic teacher	
College	University of Wroclaw

Table of Contents

Biography

He defended his doctoral thesis in 1961 and habilitated in 1967 at the University of Wrocław , where he lectured from 1974 to 2006. He served in his life many different functions at the University. head of the Institute of Mathematics , dean of the Faculty of Mathematics and Physics and vice-rector for scientific matters [2] .

In 1968 he was awarded the Stefan Banach [3]

Some work

- *The Development of prime number theory* . Springer , 2002.
- *Elementary and Analytic Theory of Algebraic Numbers* . Ed. 3. Springer, 2004. ISBN 83-01-13604-9 .
- *Number Theory* . Ed. 3. PWN Scientific Publishing House , 2003.

- *Uniform distribution of sequences of integers in residue classes* . Springer, 1984. ISBN 3-540-13872-2 .
- *Polynomial mappings* . Springer, 1995. ISBN 3-540-59435-3 .

Employment period	1974-2006

Footnotes

1. *Home page - Władysław Narkiewicz* (http://www.math.uni.wroc.pl/~narkiew/) , www.math.uni.wroc.pl [access 2017-11-27] .
2. Akademisches Kaleidoscope, page 6 (http://www.kaleidoskop.uni.wroc.pl/2006/15.pdf) (pdf)
3. Information on ptm.org.pl (http://www.ptm.org.pl/kategorie/konkursy/nagrody-glowne-ptm/nagroda-glowna-ptm-im-stefana-banacha?page=11)

External links

- Home page of Władysław Narkiewicz (http://www.math.uni.wroc.pl/~narkiew/)
- [1] (http://genealogy.math.ndsu.nodak.edu/id.php?id=133245)
- Popular (http://www.deltami.edu.pl/delta/autorzy/wladyslaw_narkiewicz/) - science articles (http://www.deltami.edu.pl/delta/autorzy/wladyslaw_narkiewicz/) in the Delta monthly

Źródło: „https://pl.wikipedia.org/w/index.php?title=Władysław_Narkiewicz&oldid=51831758"

---------- Forwarded message ---------
From: **"Władysław Narkiewicz"** <Wladyslaw.Narkiewicz@math.uni.wroc.pl>
Date: Sat, Feb 22, 2020 at 5:45 PM
Subject: Re: Open Letter Requesting Peer Review of Scientific Claim
To: P. Narasimha Reddy <nrriemann@sreenidhi.edu.in>

Dear Dr Reddy,

Thank you for sending me the information about the paper of Dr Eswaran concerning Riemann Hypothesis. I had a short look at the paper and wrote some small comments on it, which you will find in the attachment. I observed that some introductory steps can have rather simpler proofs, but noticed also that a part of the argument is unconvincing.

Yours sincerely

Wladyslaw Narkiewicz

SEE ATTACHMENT: From_Wladyslaw_22_Feb_2020_RHComment.pdf

In sections **I** and **II** I present two standard proofs of two assertions established in the paper, and in sections **III, IV** I point out some incorrect steps.

I. Re section (4) of Extended Abstract

If $\lambda(n) = (-1)^{\Omega(n)}$ *is the function of Liouville,* A *is the set of all* $n \geq 1$ *with* $\lambda(n) = 1$ *and*

$$A(x) = \#\{n \leq x : \ n \in A\},$$

then

$$A(x) = \frac{x}{2} + o(x). \tag{1}$$

Proof. Since

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)}$$

and $1 + \lambda(n)$ vanishes if $\lambda(n) = -1$ and equals 2 otherwise, we can write for $\Re s > 1$

$$\sum_{\substack{n \\ \lambda(n)=1}} \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{1+\lambda(n)}{2n^s} = \frac{1}{2}\left(\zeta(s) + \frac{\zeta(2s)}{\zeta(s)}\right) = \frac{1}{2}\frac{1}{s-1} + g(s)$$

where $g(s)$ is a function regular for $\Re s \geq 1$. It remains to observe that now one can apply Ikehara's theorem to obtain (1). □

II. Re section (5) of Extended Abstract

The sequence of values of the Liouville function $\lambda(n)$ *is not periodic.*

Proof. Assume that for sufficiently large n, say for $n > n_0$ the sequence $\lambda(n)$ is periodic with period N, thus

$$\lambda(n+N) = \lambda(n) \tag{2}$$

holds for $n > n_0$.

Choose an integer $a > n_0$ with $\lambda(a) = 1$. The equality (2) implies that if $b > a$ and $b \equiv a \bmod N$, then $\lambda(b) = 1$, hence b is not a prime, thus any prime congruent to $a \bmod N$ does not exceed n_0, and Dirichlet's theorem shows that this is possible only if $GCD(N, a) > 1$. Therefore if for an integer $c > n_0$ one has $GCD(c, N) = 1$, then $\lambda(c) = -1$. If c_1, c_2 both exceed n_0 and are co-prime to N, then their product c_1c_2 is also co-prime to N, hence

$$-1 = \lambda(c_1c_2) = \lambda(c_1)\lambda(c_2) = (-1)(-1) = 1,$$

a contradiction. □

III. Re section (6) of Extended Abstract

At the bottom of p.3 we read:

"*Specifically, non-cyclicity would preclude any dependence of the type*

$$\lambda_n = f(\lambda_{n-1}, \lambda_{n-2}, \lambda_{n-3}, \ldots, \lambda_{n-M})$$

for finite M."

This is simply untrue, as the example $\lambda(6) = \lambda(2) \cdot \lambda(3)$ shows. Since the function $\lambda(n)$ is completely multiplicative, every value of it at composite n depends on the smaller values of the function λ.

IV. Re Chapters 3–5

The author presents an argument to show that after a suitable permutation σ of summands in the series $F(s) = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}$ one obtains a series $G_\sigma(s) = \sum_{n=1}^{\infty} \frac{h(n)}{n^s}$ converging in a region to the left of the line $\Re s = 1$. The value of the sum of this series at a given point s_0 depends on the permutation σ, and a theorem of Riemann-Steinitz shows that if the series of $F(s)$ does not converge at s_0 and s_0 is not a pole of $F(s)$, then there are infinitely many complex numbers z such that with a certain permutation σ one has $G_\sigma(s_0) = z$. This shows that $G_\sigma(s)$ may have no connection to the function $F(s) = \zeta(2s)/\zeta(s)$ and the sum $\sum_{n\leq x} h(n)$ is not related to the sum $\sum_{n\leq x} \lambda(n)$.

Moreover the uniqueness of expansion of a function into a Dirichlet series implies that both equations (5.19) and (5.20) can hold only if one has $\lambda(n) = h(n)$ for every n, thus the permutation σ is the identity.

--

Dr. Kumar
Eswaran <kumar.e@gmail.com
>

to: Władysław Narkiewicz
<Wladyslaw.Narkiewicz@math.uni.wroc.pl>

cc: "P. Narasimha Reddy"
<nrriemann@sreenidhi.edu.in>

date: Feb 24, 2020, 8:46 PM

subject: Response to Your Comments on the paper on RH

mailed-by: gmail.com

Dear Dr Wladyslaw Narkiewicz,

I thank you for your thoughtful comments on the above subject, which has been forwarded to me by the Convenor for my response.
My reply to your comments and clarifications are attached.

I thank you once again for the trouble you have taken to read my paper.

With Best Wishes
Regards
Dr K. Eswaran
Professor

SEE ATTACHMENT: Feb_24_2020_Reply_one_to_Wladslaw.pdf

Reply to Dr. Wladyslaw Narkiewicz

By K. Eswaran

February 24, 2020

Dear Dr. Wladyslaw Narkiewicz,

I thank you very much for your email and the trouble you have taken to read my paper.

Regarding your comments (I) and (II):

I very much appreciate your effort to find alternative proofs, it speaks a lot for the genuine interest that you have in Mathematics, I applaud your efforts!

However, I have to study them very carefully, in order to see how they can be used to prove RH.

Now I deal with your objection **:(III)**

I had made a statement '*Specifically the non-cyclicity would preclude any dependency of the type:*

$$\lambda_n = f(\lambda_{n-1}, \lambda_{n-2}, \lambda_{n-3},....,\lambda_{n-M})$$

for finite M'

All I wanted to show is that the λ-sequence, is un-predictable. That is for very large n, it is impossible to find a fixed integer M such that if you are given the M consecutive values: $\lambda(n-1)$, $\lambda(n-2)$, $\lambda(n-3)$,...., $\lambda(n-M)$ (and no other information) it is impossible for you to predict what the next value $\lambda(n)$, will be i.e. we will not be able to say if $\lambda(n) = +1$ or if $\lambda(n) = -1$. Just like the situation when you have made very many coin tosses and you actually know the values of M consecutive coin tosses say $c(n-1)$, $c(n-2)$, $c(n-3)$,...., $c(n-M)$, but this information will not enable you to predict the next coin toss i.e. you cannot tell if $c(n) = +1$(H) or if $c(n) = -1$(T). (H≡Head, T≡Tail). I follow the method of proof adopted by Godel, the argument is as follows: If any number belonging to a sequence is predictable, from a set of M previous values (M has to be a fixed number, however large), then there exists a recursive function such as $f(\lambda_{n-1}, \lambda_{n-2}, \lambda_{n-3},...., \lambda_{n-M})$ which makes the prediction possible. If there is no such fixed M and function f then the number is unpredictable.

I think the above paragraph should clarify matters. Remember I want to show that for very large N the function L(N) behaves like a random walk.(Note I do not say that the λ-sequence is exactly equal to that of a coin toss (it is certainly not), but I show that for large N the behavior of L(N) is like a random walk and therefore the conditions of Littlewood's theorem are satisfied and RH is therefore proved.

Your second objection **(IV) i**s really the problem of notation.

To clarify I use the form (5.21) just to prove Littwood's theorem. This is done from Equations (5.22) to (5.24), thus proving Littlewoods Theorem stated in page 11, in the paragraph following (5.24). In actuality in order to study L(N) I use only the form when $\lambda(n)$ is substituted for $g(n)$or $h(n)$.This is again clarified in the last two paragraphs after (5.26b) in page 12.

In brief I use only $\lambda(n)$ to estimate L(N).

In summary, the proof of RH follows the 4 steps indicated in Ref[2], which may be followed for guidance.

-x-x- X-

I hope I have clarified all your points[1]. I will be most happy to reply to any further points. Once again I thank you for your interest.

Regards

K.Eswaran (24 February, 2020)

[1]You may wonder why I introduce Eq (1.2) in the form (3.10), when in actuality both the forms are equivalent because each term which occurs in (1.2) also occurs only once in (3.10).(in fact (3.10) the reordering of terms seems to make it unnecessarily complicated!). This is because, when I discovered Eq (3.10), I quickly realized that the L(N) behaves like a random walk. An examination of (3.10) led me to **Figure 1 and the concept of towers, which in turn lead me to the proof of RH.** As stated in the second Footnote in page 6, large part of section 3 and 4 and some of 5 can be omitted in a first reading

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"Władysław Narkiewicz" <Wladyslaw.Narkiewicz@math.uni.wroc.pl>

to: "Dr. Kumar Eswaran" <kumar.e@gmail.com>

date: Mar 2, 2020, 1:34 PM

Re: Response to Your Comments

Dear Dr. Eswaran,

Thank you for your clarifications.   I analyzed profoundly your paper in which you describe the used arguments. I did not find the proof being complete and my comments to it you will find in the attached file.

With my best wishes

Wladyslaw Narkiewicz

SEE ATTACHMENT: From_Wladyslaw_2nd_March_2020_Further_Comments.pdf

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Further comments

1. Your main idea is to use the observation that the function of Liouville resembles the coin-tossing sequence. This observation is correct, the sequence $\lambda(n)$ can be regarded as a realization of the random sequence of numbers 1 and -1, having both the probability 1/2. Based on this observation one can expect that for the sum

$$L(N) = \sum_{n=1}^{N} \lambda(n)$$

the inequality

$$|L(N) \leq c(\varepsilon) N^{1/2+\varepsilon} \tag{1}$$

holds for every $\varepsilon > 0$, showing the convergence of the series

$$f(s) = \sum_{k=1}^{\infty} \frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)}$$

in the open half-plane $\Re s > 1/2$, hence implying the regularity of $\frac{\zeta(2s)}{\zeta(s)}$ in that half-plane. Riemann's Hypothesis would be an obvious consequence.

One has only remember all assertions about infinite random sequences (in particular for the coin-tossing sequences) hold with probability 1, i.e., not for all sequences but only for almost all such sequences. Hence if one studies a particular realization, then each needed property of random sequences must get a proof. Arguments *per analogiam* are not permitted.

Similarity of some properties of the sequence $\lambda(n)$ to properties of random sequences has been studied in the last years by several authors. Usually one says that $\lambda(n)$ is a pseudo-random sequence. There are several open questions concerning this similarity. For example, if $a(n)$ is a random sequence consisting of numbers ± 1, then for every k there is a positive probability for the set of numbers m such that

$$a(m + j) = \varepsilon_j$$

happens for $j = 1, 2, \ldots, k$, where $\varepsilon_j \in \{-1.1\}$ is given. It has been conjectured by S.Chowla in 1965 that also the sequence $\lambda(n)$ has this property, but this has been established only for $k = 3$ by A.Hildebrand in 1986 (Math.Proc.Cambridge Math. Soc. (1986), 229–236), and the probabilities in this case were recently determined by K.Matmäki and M.Radziwiłł (Forum Math. Sigma 4 (2016), e14, 1–44).

This indicates that not every property of the coin-tossing sequence transfers automatically to the sequence $\lambda(n)$, and this applies in particular to the assertion $|L(N)| \leq c\sqrt{N}$, used in your proof of RH, which is an analogue of the true corresponding assertion in the case of coin-tossing.

A similar attack on the Riemann Hypothesis using the Moebius function $\mu(n)$ has been tried by T.J.Stieltjes (Comptes Rendus Acad. Sci. Paris 10 (1985), 153–154) in 1885, who asserted the convergence of the series

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^x}$$

for all $x > 1/2$. He did not publish a proof and nothing about it has been found in his preserved notes, but in a letter to E.Hermite he wrote that the proof is based on the inequality

$$|M(N)| = \left| \sum_{n=1}^{N} \mu(n) \right| \leq B\sqrt{N}$$

with a suitable constant B.

2. On page 2 of your paper you want to show that the sequence $\lambda(n)$ behaves like a sequence of coin tosses and state two its properties, equal probabilities and non-periodicity and independence.

a) **Equal Probabilities**.

I understand that you mean by that the following correct equalities concerning $A^+(N) = \#\{n \le N : \lambda(n) = 1\}$ and $A^+(N) = \#\{n \le N : \ \lambda(n) = -1\}$:

$$\lim_{N\to\infty} \frac{A^+(N)}{N} = \lim_{N\to\infty} \frac{A^-(N)}{N} = \frac{1}{2}. \tag{2}$$

This is actually an assertion about the density of the sets $A^+(N)$ and $A^-(N)$ and to be able to interpret it in probabilistic language one has to define the underlying probability space. Unfortunately, there is not possible to define a probability space on the set of positive integers in which every number has the same probability, and to go around this trouble one usually considers probability on the set of the first N positive integers, with every its element acquiring probability $1/N$. Then one applies the existing probabilistic tools to study the considered problem, and looks what happens when N tends to infinity. Sometimes this leads to good results (see e.g. the two-volume book by Peter Elliott [”Probabilistic Number Theory”, Springer 1979]).

You write on p.23 that the equalities (2) are consequences of Theorem 1 on that page, but this is incorrect. Theorem 1 states only that there exists a bi-unique correspondence $A^+ \Longleftrightarrow A^-$ between the sets $A^+ = \#\{n : \ \lambda(n) = 1\}$ and $A^- = \#\{n : \ \lambda(n) = -1\}$, hence they have the same cardinality, but this happens for all pairs of infinite subsets of positive integers, and if this theorem would imply (2), then it would also imply the following completely false equalities:

If $\pi(N)$ denotes the number of primes $p \le N$, and $X(N)$ denotes the number of composite numbers $\le N$, then

$$\lim_{N\to\infty} \frac{\pi(N)}{N} = \lim_{N\to\infty} \frac{X(N)}{N} = \frac{1}{2},$$

whereas the first limit actually equals 0 and the second equals 1.

The equalities (2) are not new, being equivalent to

$$\lim_{N\to\infty} \frac{1}{N}\#\{n \le x : \ 2|\Omega(n)\} = \lim_{N\to\infty} \frac{1}{N}\#\{n \le x : \ 2 \nmid \Omega(n)\} = \frac{1}{2}, \tag{3}$$

whose proof has been indicated already in 1898 by H. von Mangoldt, and details were given in §167 of the book of E.Landau (”Handbuch der Lehre von der Verteilung der Primzahlen”), published in 1909. Another proof, based on a tauberian theorem of Ikehara, I showed you in my previous message.

b) **Non-periodicity and Independence**

Your analytical proof given in Appendix III that the sequence $\lambda(n)$ is non-cyclic is correct.

There is a problem with the notion of independence. On p.3 you define the dependence of values of the function λ writing: ”... *$\lambda(n)$ is dependent on $\lambda(m)$, if the latter is required to find the former*”, but this has no mathematical meaning. Here $\lambda(n)$ is a well-defined function for every positive integer n and the whole sequence of its values at integers is well-defined from the very beginning. This makes the situation essentially differing from a coin-tossing sequence whose elements are created consecutively.

On the bottom of p. 2 you wrote ”*Specifically, non-cyclicity would preclude any dependence of the type*

$$\lambda_n = f(\lambda_{n-1}, \lambda_{n-2}, \lambda_{n-3}, \ldots, \lambda_{n-M}) \tag{4}$$

for finite M.”

This is not correct, just look at the Fibonacci sequence defined by $F_1 = F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$. It tends to infinity, hence is non-cyclic, but satisfies a relation of type (4) with $f(X, Y) = X+Y$. Non-existence of a relation of type (4) for a particular function, say $X(n)$, shows only that the sequence $X(1), X(2), \ldots$ is not a recurrence sequence.

You use later in Appendix IV your assertions discussed above in a) and b) to claim that $L(N)$ series is a random walk and deduce in Theorem 4 the bound on $L(N)$ needed in the proof of RH, but as I noted above, the assertion b) has no correct proof. The discussion of independence in Appendix IV lacks mathematical precision and the reader can wonder how this discussion implies that the sequence $\lambda(n)$ is equivalent (what kind of equivalence is here used?) to coin tosses, as indicated in the title of the Appendix.

--

Dr. Kumar
Eswaran <kumar.e@gmail.com
>

to: Władysław Narkiewicz
<Wladyslaw.Narkiewicz@math.uni.wroc.pl>

date: Mar 5, 2020, 7:48 AM

subject: My Response to your Further Comments

mailed-by: gmail.com

Dear Professor Wladyslaw Narkiewicz,

I thank you very much for your detailed reading and comments. I am truly grateful for the time that you are spending and I am very much indebted to you for this!

Please find my response in the first attachment.
Thank You Once again.
Regards
K. Eswaran
P.S. This email contains two other attachments, for your convenience.

SEE ATTACHMENTS: (1) March_5_2020_reply_Two_Wladyslaw.pdf

(2) Pathway_to_RH_lecture_by_Eswaran_IIT_Madras.pdf

(3) Summary_of_lecture_IIT_Madras.pdf

Dear Professor Wladyslaw Narkiewicz,

I thank you very much for your detailed reading and comments. I am truly grateful for the time that you are spending and I am very much indebted to you for this!

Firstly, I reply to the points you have made in page 2 that are of immediate relevance to my proof of RH.

(a) Equal Probabilities

Since by definition $\lambda(n) = (-1)^{\Omega(n)}$, where $\Omega(n)$ is the number of prime factors of n, multiplicities included, with the definition $\lambda(1) = 1$. Hence, $\lambda(n) = -1$,not only when n is a prime number but also if n is a composite number with odd number of prime factors, e.g. 12= 2x2x3, hence $\lambda(12) = -1$. If we denote the number of composite integers n less than or equal to N having odd number of prime factors i.e.($\lambda(n) = -1$),as $X_-(N)$ and the number of composite integers n less than or equal to N having even number of prime factors i.e.($\lambda(n) = 1$), as $X_+(N)$. Therefore, the equality that you wrote down should actually read as:

$$lim(N \to \infty)\frac{\pi(N) + X_-(N)}{N} = lim(N \to \infty)\frac{X_+(N)}{N} = \frac{1}{2}$$

there is no contradiction.

In the proof of equal probabilities, I purposely avoid falling into the trap of using Cantor's mapping technique. I label every integer by a unique triad of integers. Then I sort all the integers into sets called "Towers', each integer in a Tower is followed by an unique (higher) integer in the same tower and their λ-values alternate as + 1 then -1, then +1 etc. Further, every integer occurs only once in some tower or other and I have further ensured that for the special integers say m, which occur at the base of any Tower, I have arranged matters such that if for every integer m at the base which has $\lambda(m) = 1$ there is another unique integer m' at the base of another Tower which has $\lambda(m') = -1$ and vice-versa. All these considerations, in my opinion, make the proof of equal probabilities for $(N \to \infty)$ water tight.

I have given another Alternative Proof for equal probailities. The only assumption made in this second proof is that in the set of all ordered integers every odd integer is followed by an even integer and precedes another odd integer -thus making the number of even integers equal to the number of odd integers. The second proof is by construction and therefore avoids mappings. The link to the Alternative proof is given as Ref[3) in the first Invitation Email Sent to you.

(b) Non-periodicity and Independence

The Fibonacci series is not the right analogy because every term in the series takes newer and newer integer values and there is almost no repetitions except for the 2nd and 3rd terms. But the $\lambda-$ series is extremely constrained, each term can take either $+1$ or -1.

It is because of this reason that for some large N_0 all the $\lambda(n)$, (with $n > N_0$), cannot be predicted by any formula of the form:

$$\lambda(n) = f(\lambda(n-1), \lambda(n-2), \lambda(n-3), ..., \lambda(n-M))$$

where M, is a fixed integer, this is because the function f takes as inputs M consecutive values of $\lambda's$ and evaluates to +1 or -1 to predict the next λ viz $\lambda(n)$. But these M consecutive values of $\lambda's$ can be thought of as a string of M numbers, but since each number can take only the values +1 or -1, there can be only 2^M different patterns of strings each of length M. Hence this means the above formula will at most predict a sequence of $\lambda's$ which has a maximum length of $L = 2^M$ and after this the entire predictions will repeat and thus form a cycle. [QED}

I enclose as an attachment to this Email,(File called ESWARAN LG FULL LECTURE) my Invited Lecture to IIT Madras which gives more examples please see slide 57. The other slides of the lecture can also be consulted for the derivation of Equal Probabilities and other matters, including Alternative Proofs (see slides 69-70)of several theorems I have proved in the Main paper.

My General Comments and Closure

The Extended Abstract Ref[2] in the first Email describes the crucial Steps used to provide the proof of RH. For your convenience, I have attached this as a file called SUMMARY of LECTURE IIT

The basic idea in the proof is to show that for very large N (actually at infinity), L(N) behaves like a random walk. This is the requirement laid out by Littlewood's Theorem to prove RH affirmatively. Now we know that the $\lambda-$series is deterministic and of infinite length, so it strictly is not exactly obtainable by coin tosses. However, for very large N it so happens that the deterministic nature of the $\lambda's$ do not effect the essential statistical similarities between the $\lambda-$series and a series of coin tosses. In a random walk it has been proved that the root mean square distance in N steps has a square root behavior (viz $\sqrt{N}$) for large N. The derivation for this expression has been done (example see pages 3-5, in Stochastic Problems in Physics and Astronomy by S. Chandrasekar Rev. of Modern Physics (1943) vol. 15, no. 1), by assuming that each step (like coin tosses) have (i) Equal probabilities of being (+1) or (-1), (ii) Independence (iii) unpredictability. So any sequence which has all these properties must have the same square root behavior for their root mean square value, it so turns out that for large N the $\lambda-$series has all these three properties and therefore $L(N)$ will also have the square-root behavior, proving RH.

I hope I have clarified the points you have raised.

I will be ever very grateful for the time you have taken. It has been a pleasure to read and think about your insightful comments.

Regards K. Eswaran 5th March 2020

"Władysław Narkiewicz" <Wladyslaw.Narkiewicz@math.uni.wroc.pl>

to: "Dr. Kumar Eswaran" <kumar.e@gmail.com>

date: Mar 9, 2020, 4:51 PM

subject: Re: My Response to your Further Comments

Dear Professor Eswaran,

I looked again thoroughly at your text and your explications. I subsumed my thoughts about your paper in the attached file. Unfortunately there is a deep hole in your argumentation, when you assert without any proof that a theorem in the theory of random sequences implies a bound for the sums of values of the function $\lambda(n)$.

Do not worry about this situation. Several excellent mathematicians tried without success to prove Riemann Hypothesis.

With every good wish

Wladyslaw Narkiewicz

SEE ATTACHMENT: From_Wladyslaw_9th_March_2020_Final_Comments.pdf

1. The central point in your method is the proof of the inequality

$$|L(N)| = \left|\sum_{n=1}^{N} \lambda(n)\right| \le BN^{1/2+\varepsilon} \tag{1}$$

for every positive ε, with the constant B depending on ε. This is actually an easy observation that (1) implies Riemann Hypothesis, because by elementary partial summation one deduces from (1) the convergence of the series

$$f(s) = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^x} \tag{2}$$

for all real $x > 1/2$, and by the theory of Dirichlet series this implies the convergence of that series in the open half-plane $\Re s > 1/2$ to a regular function. Since for $\Re s > 1$ the sum of this series equals $\zeta(2s)/\zeta(s)$, we get $f(s) = \zeta(2s)/\zeta(s)$ also in that half-plane and it follows that $\zeta(2s)/\zeta(s)$ does not have poles in $\Re s > 1/2$, hence Riemann Hypothesis follows. The same argument works also for the sum $M(N) = \sum_{n\le N} \mu(n)$ and in that case occurs already in the failed attempt by Stieltjes.

The main problem with your approach to RH lies in the proof of (1).

2. At the end of your last message you wrote that any sequence which has the following properties:

(i) Equal probabilities of being 1 *or* -1*,*
(ii) Independence,
(iii) Unpredictability,

must have the same behavior as a random walk, and applying this to the sequence $\lambda(n)$ you infer the inequality (1).

On your slide 57 you define the notion of independence, which shows that you call elements of a sequence $a_1, a_2, \dots$ independent if the sequence is not a recurrent sequence. Unfortunately you do not give a precise definition of unpredictability. Mathematics is a formal science, and every notion occurring in a mathematical argument must have a proper formal definition. I understand that you use the word "unpredictable" in the common sense, but this makes your argumentation incomplete.

I did not find in your paper a proof of the assertion that the conditions (i), (ii) and (iii) (whatever they mean) imply for the sequence $\lambda(n)$ the inequality (1). I presume that you had in mind an application of the following well-known theorem about simple random walk:

THEOREM. *Let $X_1, X_2, \dots$ be an infinite sequence of independent random variables with values ± 1, each of probability $1/2$. If S_N denotes the random variable $\sum_{j=1}^{N} X_j$, and $E(|S_N|)$ denotes the expected value of the random variable $|S_N|$, then the limit*

$$\lim_{N\to\infty} \frac{E(|S_N|)}{N}$$

exists and is equal to $\sqrt{2/\pi} = 0.7978\dots$.

Unfortunately, nowhere in your paper you mention how one can formally deduce your assertion about (1) from this well-known theorem. I wonder whether such a proof is possible, as the elements of the sequence $\lambda(n)$ depend on n, and elements in a random sequence do not have that property.

This makes it clear that the presented proof of Riemann Hypothesis is not complete.

3. In my previous message I observed that your way of establishing the "equal probabilities" property for the sequence of values of the λ-function can lead to false result. You answered that the equality which I wrote in that message should look in another way, avoiding contradiction. It seems that I presented my argument in a not sufficiently precise way. This time I will try to do better.

Let P denote the set of primes arranged as a sequence $2 = p_1 < p_2 < p_3 \cdots$, and let $C:\ 1 = c_1 < 4 = c_2 < c_3 < \cdots$ be the set of all remaining positive integers, also arrangd in a sequence. Now I apply your own argument from your last message to obtain the proof that the conditions "an integer n lies in P" and "an integer n lies in C" have "equal probabilities". Note that every positive integer lies either in P or in C and the sets P, C have no elements in common. With every prime p_n I associate the unique number c_n lying in C. And now I copy your argument in this case:

"I have arranged matters such that for every integer m in P there is another unique integer m' in C and vice-versa. This makes the proof of equal probabilities water tight."

Now observe what happens. Denote by $\pi(N)$ the number of primes $p_n \leq N$, and by $C(N)$ the number of numbers $c_n \leq N$.

The equal probabilities property implies

$$\lim_{N\to\infty} \frac{\pi(N)}{N} = \lim_{N\to\infty} \frac{c(N)}{N} = \frac{1}{2},$$

but this is not true, as actually one has

$$\lim_{N\to\infty} \frac{\pi(N)}{N} = 0.$$

All this is of minor importance, as the result which you wanted to prove is true, found by von Mangoldt already in the 19th century.

On Fri, Mar 13, 2020 at 7:57 AM Dr. Kumar Eswaran <kumar.e@gmail.com> wrote:

Dear Wladyslaw Narkiewicz,

I thank you very much for your detailed reading and comments. It is my very great good fortune to have a learned person like you to be reading my papers.

I agree with you that I must give proper Definitions in a Mathematical paper.

I am following your advise. At the moment, I am in the process of framing my arguments and basing them on properly defined entities and terminologies.

This is taking me some time. I will get back to you in 5 or 6 days.

Regards
K.Eswaran
- - - - - - - - -

Dr. Kumar Eswaran <kumar.e@gmail.com>

to: Władysław Narkiewicz <Wladyslaw.Narkiewicz@math.uni.wroc.pl>

date: Mar 17, 2020, 2:29 PM

subject: My Reply to your latest comments of March 9th 2020

Dear Wladyslaw Narkiewicz,

I want to once again express my thanks to you for sparing so much of your time reading my work. I do not know how to express my gratitude except by taking every word of yours seriously (thus demonstrating my profound respect for you) and try to make a complete clarification to the best of my ability.

Please find my detailed response to your queries in the attached.

I will be most happy to clear any further doubts (if any).

God Bless!
Regards
Kumar Eswaran

Please See Attachment,Thank You!

See ATTACHMENT: March 17-2020_Reply_three_to_Wladyslaw.pdf

Letter to Prof Wladyslaw Narkiewicz

Dear Professor Wladyislaw Narkiewicz,

I now define predictability of the $\lambda' s$ follows.

DEFINITION: We say that $\lambda(n)$ is predictable if, there exists a finite integer M, such that for every $n > N_0$, the $\lambda(n)$, is derivable from its M previous values $\lambda(n-1), \lambda(n-2), \lambda(n-3), ...\lambda(n-M)$.

Note, the value of n is not explicitly known: only the values $\{\lambda(n-r), r = 1, 2, ..M\}$ are known.

We then assume that if $\lambda(n)$ is not predictable then it is independent.

Of course, if $\lambda(n)$, is derivable as above, then there exists some function f s.t :

$$\lambda(n) = f(\lambda(n-1), \lambda(n-2), \lambda(n-3), ...\lambda(n-M))$$

The justification for these definitions are given in Section 1 and Section 2.

SECTION 1:

Regarding your objection to the treatment of Eq(1) viz.

$$|L(N)| = \Sigma_{n=1}^{N} \lambda(n) \tag{1}$$

which by Littlewood's Theorem is supposed to satisfy the relationship:

$$|L(N) \leq C\, N^{a+\varepsilon} \tag{2}$$

for large N. In (2) I have used a in the exponent on the R.H.S. instead of 1/2, according to Littlewood if it so happens that $a = 1/2$. then RH is proved.

Comparing the above with the summation of a sequence "coin tosses" (or random walk) depicted by:

$$|S_N| = \Sigma_{j=1}^{N} X_j \tag{3}$$

where the random variable $X_j$takes on values $+1$ or -1, with equal probability and are independent of each other then, it is well known that for large N:

$$|S_N| \approx \sqrt{\frac{2}{\pi}}\, \sqrt{N} \tag{4}$$

Now, we have already shown that the $\lambda(n)$ has the probability of being $+1$ or -1 for very large n. In addition I have proved (in Appendix IV) that for large n, there is no way of knowing $\lambda(n+1)$ if we happen to know $\lambda(n)$.[1]This was sufficient to prove "independence" for large n. (Note if we know $\lambda(n)$ the next predictable value is $\lambda(2n) = -\lambda(n)$, and $2n$ is very far away from from n for large n.) Later, I used another criterion of "independence" surmising that if

[1]In fact, in Appendix IV, we have an alternative proof of 'independence' because we showed that the sequence of $\lambda' s$ between $\lambda(n+1)$ and $\lambda(2n)$ are independent of each other; this length of this strip becomes infinite as n tends to infinity.

$\lambda(n+1)$ is derivable (predictable) from its M previous values, then the $\lambda(n)'s$ are predictable and hence not independent. But I showed that no fixed finite M exists[2] and hence we can consider the $\lambda's$ are independent for large n. (Therefore, this second criterion is more general because the first corresponds to M=1). We show in the second Section 2 that this definition of predictability is the correct one because by proving the $\lambda's$ are un-predictable (by this definition) and hence independent is able to explain the behavior ('phenomena') of the $\lambda's$ over large consecutive sections.

Hence we see that by using the same logic which derives (4) from (3) we can derive (2) from (1) and we will have the position of the critical vertical line at $a = 1/2$ thus proving R.H. However, in the paper I use a yet more rigorous analysis (see last paragraph of page 14 of the main paper) which was done in a study on independent random variables done by Khechin and Kolmogorov using iterative logarithms, their analysis gives the dependence of ε on N and how as N tends to infinity. From their result I have shown that the "width" of the critical line tends to zero, proving that all the nontrivial poles of F(s) and hence the zeros of $\zeta(s)$ lie on the critical line. QED

has

SECTION 2 : Phenomenological study of behavior of $\lambda(n)'s$

In Section2 of Appendix VI of the Main paper, I have calculated consecutive $\lambda(n)'s$ forming a large sequences of length $M$of the form $\Lambda[N_0, M] \equiv \{\lambda(N_0),\ \lambda(N_0+1),\ \lambda(N_0+2), ...,\ \lambda(N_0+M-1)\}$.where $N_0$is some large integer. And for each such case I did a χ^2 fit which compare sit with a Binomial sequence (coin tosses) of the same length M. as described by Knuth . In each case the value of $\chi^2 \leq 4.0$ showing that the $\lambda's$ in the sequence are indistinguishable from coin tosses. In the Tables given in the Appendix VI , I typically choose $N_0 = N$ (a perfect square and $M = \sqrt{N}$. E.G. the items in row 2 Table 1.4 show that the sequence of length $M = 100,000$ starting from $\lambda(10,000,000,001)$,......,to $\lambda(10,000,100,000)$ has a χ^2value equal to 1.15,thus it is statistically indistinguishable from a sequence of $100,000$ coin tosses or a random walk of $100,000$steps. This "phenomena' depicting the statistical behavior of the $\lambda's$ happens for all the sequences of ALL the entries in the Tables in Appendix VI. The reason for this phenomena is because each λ in the set $\Lambda(N, \sqrt{N})$ is sampled from a different 'Tower' and computing the summatory function $L(N, \sqrt{N})$ is like randomly picking a number from the next Tower which then may have value +1(H) or -1(T) with equal probability.[3]

Since all the sequences $\Lambda[N_0, M]$, of differing values of $N_0$and M, behave like coin tosses regardless of their starting value N_0, we can say that the statistical behavior does not depend on the starting value N_0 ,further the statistics of

[2]If no fixed finite M exists (i.e. M grows large with n or is infinite) it automatically means that the $\lambda(n+1)$ is all the more unpredictable from $\lambda(n)$ thus the $\lambda's$ are practically independent for large n.

[3]It is possible to argue similarly, and demonstrate that the sequence $\Lambda(N_0, M)$, for large N_0 and $M < 2N_0$will also statistically behave like coin tosses because each λ in the sequence will be drawn from a different Tower. and could be randomly +1(H) or -1(T).

$\Lambda[N_0+k,M]$ (k being a small number) is the same as that of $\Lambda[N_0,M]$ we call this property as "Translational Invariance" which is certainly true if we are talking of the N_0^{th}coin toss $c(N_0)(\equiv X_{N_0})$; the coin tosses are all unpredictable and therefore independent. Now the numerous computed χ^2 values show that the sequences of the $\lambda's$ are statistically indistinguishable and also have the property of being statistically "Translational Invariant" for their corresponding sequences of coin tosses. Hence, to actually explain this phenomena of the statistical behavior we must show that the $\lambda's$ are also statistically "Translational Invariant", 'unpredictable' and therefore 'independent'. Hence we are *by tour-de-force* led to the following definitions:

DEFINITION: We say that $\lambda(n)$ is predictable if, there exists a finite integer M, such that for every $n > N_0$, the $\lambda(n)$, is derivable from its M previous values $\lambda(n-1), \lambda(n-2), \lambda(n-3), ...\lambda(n-M)$.

Note, the value of n is not explicitly known only the values $\{\lambda(n-r), r = 1, 2, ..M\}$ are known.

We then assume that if $\lambda(n)$ is not predictable then it is independent.

Of course, if $\lambda(n)$, is derivable as above then there exists some function f s.t :

$$\lambda(n) = f(\lambda(n-1), \lambda(n-2), \lambda(n-3), ...\lambda(n-M))$$

In the Main Paper we proved that such a function cannot exist otherwise it will make the λ-sequence cyclic, proving un-predictability and hence independence of the $\lambda's$.

From the above analysis we can conclude that as N tends to infinity the statistical behavior of the $\lambda's$ and the 'coin tosses' become very similar. Thus just as Eq.(4) follows from Eq. (3) for coin tosses, we are forced to conclude that Eq. (2) (with a=1/2) follows from Eq. (1) for the $\lambda's$.[4]

We have thus shown that the λ-series has a dominant statistical behavior of a random walk and therefore the summatory function $L(N)$ can be computed as is done in the paper using the iterative logarithm formula derived by Khechin and Kolmogorov, thus proving R.H.

It cannot be over emphasised that apart from proving RH, I have given reasons for the phenomena of large subsets sets of consecutive $\lambda's$ denoted as $S_+(N) = \Lambda(N+1, \sqrt{N})$ and $S_-(N) = \Lambda(N-\sqrt{N}+1, \sqrt{N})$, (Na square integer) behaving like coin tosses, (see Eqs. (13) and (14), Sec. 4, Appendix VI of Main paper). This is important because by taking a collection of all perfect squares N, the sets $S_+(N)$and $S_-(N)$ contain all the $\lambda's$ upto infinity. Since,

[4]This is because the deterministic formula $\lambda(p.n) = \lambda(p).\lambda(n) = -\lambda(n)$, which can be used to predict the other values of the $\lambda's$ having known $\lambda(n)$ does not disturb the statistics of any large sequence, for example $\Lambda[N, \sqrt{N}]$ (N being square integer), this is because even if $\lambda(n)$ belongs to the sequence, even the very next predictable value $\lambda(2n)$ lies outside the range of the sequence and does not occur in $\Lambda[N, \sqrt{N}]$. The predictive power of the deterministic formula for $\lambda(m.n) = \lambda(m).\lambda(n)$, causes only a minor perturbation on the dominant statistical 'random walk' behavior of L(N), for very large N, and whose effect fades away at infinity.

In my project log in my Researchgate webpage, I have examined the behaviour of $\lambda's$ and its effect on L(N) and they have confirmed this assertion.

the Tables given in Appendix VI are actual computed values of the $\lambda's$ we have not only proved RH but have given an explanation of the behaviour of the $\lambda's$. Therefore, I believe that: Even if one is hard pressed to deny this proof of RH, then he/she is compelled to deny the very existence of this phenomena, but the latter cannot be denied because too many computations and χ^2comparisions have been confirmed. It is because of these reasons (and of course my own intution), I humbly believe that, what we have is a proof of RH!

In this reply I have confined myself to only your objections raised as points 1 and 2. I am leaving out point 3, as you have said a proof by von Mangoldt already exists. (I have given (in Ref 3, in the 1st Invitation Letter (email) dated 3rd Feb 2020, an alternative proof which relies on induction and not mapping, but I acknowledge that it has now become academic and redundant!).

To conclude, Professor Wladyslaw Narkiewicz, I want to once again express my thanks to you for sparing so much of your time reading my work. I do not know how to express my gratitude except by taking every word of yours seriously (thus demonstrating my profound respect for you) and try to make a complete clarification to the best of my ability. I will be most happy to clear any further doubts (if any). God Bless!

Regards
Kumar Eswaran
March 17, 2020
—

"Władysław Narkiewicz" <Wladyslaw.Narkiewicz@math.uni.wroc.pl>

to: "Dr. Kumar Eswaran" <kumar.e@gmail.com>

date: Mar 30, 2020, 8:08 PM

subject: Riemann Hypothesis

mailed-by: math.uni.wroc.pl

Dear Professor Eswaran,

In my previous messages I tried to show you that your arguments are not very convincing. Your proof consists of the following steps:

1. You define predictability of a sequence in the following way:

The sequence a_n is predictable if for some integer M the term a_n (for large n) depends on M precedent terms.

This definition coincides with the definition of a recurrent sequence, used in several parts of mathematics.

2. You state that a sequence which is not predictable is independent, without defining the word "independent".

3. You recall a theorem from probability theory describing asymptotical behavior of the expected value of a series of independent random variables. I am not quite sure whether you quoted it correctly. I now it only for the case when the random variables attain the values 0 and 1. In the case when you have 1 and -1 it seems that the expected value would tend to zero. But I am not an expert in probability.

4. You state that this theorem implies that if a sequence is unpredictable (in the above sense) and attains the values 1 and -1 with the same frequency, then the sum of its first N values is asymptotic to $c\sqrt N$ with some c.

5. Applying this to the sequence $\lambda(n)$ one obtains the Riemann Hypothesis, as it is well-known that if for all large N one has
$$\sum_{n=0}^N\lambda(n)\le c\sqrt N,\eqno(*)$$
then RH follows.

You do did not a proof the step 4, and actually the assertion stated there is incorrect, as one can easily construct a sequence a_n with $a_n=1 or -1, which is unpredictable in the sense of 1, attains its values with the same frequency but does not satisfy the assertion (*).
Example:

For $k=1,2,\dots$ denote by I_k the interval $[(k-1)\cdot10^4,k\cdot10^4-1]$ and denote by A_k the set of all even integers in I_k to which $k^{3/4}\cdot10^3$ odd integers from I_k, taken at random, are added, and let B_k be the set of all remaining integers from I_k. Let A be the union of all sets A_k and B the union of all sets B_k, and define $A(x), B(x)$ to be the numbers o elements of the sets A,B in the interval $[1,x]$.

Define a function $f(n)$ by $f(n)=1$ for $n\in A$ and $f(n)=-1$ for $n\in B$, and observe that because of the randomly added elements to B the function f satisfies your unpredictability condition.

One sees immediately that we have

$$\lim_{x\to\infty}A(x)/x =\lim_{x\to\infty}B(x)/x = 1/2,$$

and
$$\sum_{n\le k\cdot10^4}f(n) = A(k\cdot10^4} - B(k\cdot10^4)) = k^{3/4}\cdot10^3,\eqno(**)$$
but according to your assertion 4 this sum should be bounded by
$$c\sqrt{k\cdot10^4} = c\sqrt k\cdot10^2,$$
in contradiction to (**).

Therefore your proof is not correct.

Do not worry, your are not the first person who presented an incorrect proof of RH. At their list one finds some excellent mathematicians.

With every good wish

Wladyslaw Narkiewicz

Dr. Kumar Eswaran <kumar.e@gmail.com
>

to: Władysław Narkiewicz
<Wladyslaw.Narkiewicz@math.uni.wroc.pl>

date: Apr 2, 2020, 8:34 PM

subject: Response to your Email of the 30th March
Re: Riemann Hypothesis

Dear Professor Wladyslaw Narkiewicz,

I thank you for your email of the 30th March; I wanted to immediately respond by pointing out that you had overlooked a crucial criteria which needs to be satisfied by your Example but it does not, thus invalidating it from being a valid contradiction. However, since you have been taking so much of interest and trouble in studying my work, I felt that a short immediate response will be discourteous and disrespectful from my side. So I decided that I would write out in fair detail the concepts, the logical sequence of thoughts and the sequence of findings which enabled me to come up with this purported proof of RH. All this took time and therefore this delay in my response.

Yes, the proof involves many areas: statistics, randomness, complex functions, concepts from computer science and information theory (Knuth, Shannon) that I borrowed apart from number theory. A proper understanding of the proof and of the methods employed would need a fair understanding of most of the above.

I plead for your patience in the perusal of the write-up (attached) and I hope it will be able to clarify much of what I have done and it will be able to clear any misgiving that you may have.

I will be ever grateful for your interest, Thank You!

Kind Regards

Kumar Eswaran

This Email is in response to your Email:
On Mon, Mar 30, 2020 at 8:08 PM "Władysław Narkiewicz"
<Wladyslaw.Narkiewicz@math.uni.wroc.pl>

SEE ATTACHMENT: April_2_2020_Reply_Four_to_Wladyslaw.pdf

- ---------

MY WRITE UP IN RESPONSE TO YOUR EMAIL of 30th March 2020

2nd April 2020

Dear Professor Wladyslaw Narkiewicz,

I thank you for the trouble you are taking over my papers.

At the outset I wish to say that the summatory function $L(N)$:

$$|L(N)| = | \Sigma_{n=1}^{N} \lambda(n) | \tag{1}$$

and the summation of a sequence "coin tosses" (or random walk) depicted by:

$$|S_N| = | \Sigma_{j=1}^{N} X_j | \tag{2}$$

(where the random variable $X_j$takes on values $+1$ or -1, with equal probability), will behave statistically identically for large N, if ALL the following conditions are satisfied by $\lambda(n)$ viz. (i) Equal probabilities, (ii) independence and (iii) unpredictability (see Extended Abstract for details). **If all these conditions are satisfied then only we can conclude that,** for large N:

$$|L(N)| \leq C\, N^{1/2+\varepsilon} \tag{3}$$

However, in constructing your Example (refer to your para 5) for $\sum f(n)$, you have overlooked one crucial requirement. The sequence $f(n)$ which you constructed though you say is unpredictable has the property that if n is even then not only $f(n) = 1$ but $f(n+2) = 1$ so one cannot say $f(n+2)$ is not deducible from $f(n)$. The sequence $f(n)$ will invariably fail the χ^2 test because most of the times $f(n)$ and $f(n+2)$ are highly correlated; the test of independence will fail. All the conditions (i), (ii) and (iii) must be satisfied, but this has not happened and because of this your result is not a valid counter example. I repeat any sequence to resemble a sequence of coin tosses MUST satisy all the three criteria: (i), (ii) and (iii). In other words, for a valid comparision, one cannot take any arbitrary sequence, of +1's and -1's, because one needs to show equal probability and unpredictability and independence before one can even begin to consider the sequence as an appropriate counter example. The reference of D. Knuth gives details of how such χ^2 tests are done. In the write-up below, I have described the results of extensive and very many computations over large streches of consecutive $\lambda's$ the χ^2test has always passed, thus placing the matter beyond reasonable dispute.

Before answering your other queries, I wish to say that if there is a random variable X_j which takes on values $+1$ or -1, with equal probability and is independent i.e. they satisy the three criterion: viz. (i) Equal probabilities, (ii) independence and (iii) unpredictability then, it can be proved that, for large N,

$$|S_N| \approx \sqrt{\frac{2}{\pi}}\,\sqrt{N} \tag{4}$$

.The proof of Eq4. , following from the definitions (2) and the three criteria (i), (ii) and (iii), is very well known result in the Statistical theory of random variables pertaining to coin tosses or 1-d random walks.See references in Footnote:[1]

Therefore, any other sequence consisting of numbers +1 or -1, which satisfies the above three criteria (i), (ii) and (iii), will have its sum of N terms behave like (4). For convenience I call this as the "Principle of Statistical Analogy."[2]Hence, by this reasoning Eq. (3) will follow from Eq. (1), because we have proved that the $\lambda(n)$ satisfies the three criteria (i), (ii) and (iii). Thus RH necessarily follows from Littlewood's theorem. **Additional Note:** Littlewood's Theorem only requires that the three criteria need only be satisfied by the λ-sequence for large N and we have proved that this is so.

Coming to your other queries: In my previous email I gave reasons for the "natural" definitions of un-predictability, independence, non cyclicity, and Equal probabiliy. For the latter situation viz equal probability, I define probability in the somewhat common sense way that is: given a randomly chosen integer n,then $\lambda(n)$ has an equal probability of having a value +1 or -1. Since $\lambda(n)$ is defined over all integers and can have only two possible values (+1 or -1) there is no need to build an elaborate machinery of probability spaces, Borel sets, Lebesgue measures etc.

All the definitions adopted by me has the virtue of being natural definitions and simple and sufficient for my purposes: which was to show that the three criteria is satisfied by the λ-sequence.

————-

Dear Professor, As far as I can tell, I have replied to almost all your queries.

Now, I wish to explain in my own way of thinking, my surmise that the RH is true. In order that you will appreciate what I say I now request you to think from my point of view i.e. the point of view of a Theoretical Physicist,[3] which is some what different form the point of view of a strict Number Theorist as I will presently explain. I request you to read the rest of this email and try to empathise with my point of view.

In mathematics a Mathematician starts from some initial premises which are axioms e.g. a geometer starts from (say) Euclid's Axioms and a Number

[1]See e.g. (1) Chandrasekhar S., (1943)'Stochastic Problems in Physics and Astronomy',Rev.of Modern Phys. vol 15, no 1, pp1-87,

(2) Khinchine, A. (1924) \Uber einen Satz der Wahrscheinlichkeitsrechnung", Fundamenta Mathematicae 6, pp. 9-20,

(3) Kolmogorov, A., (1929) \Uber das Gesetz des iterierten Logarithmus". Mathematische Annalen, 101: 126-135. (At the DigitalisierungsZentrum web site)). Also see: https:en.wikipedia.orgwikiLaw of the iterated logarithm.

(4) One dimensional Random Walk https://mathworld.wolfram.com/RandomWalk1-Dimensional.html ▷

[2]The principle of Analogy has been used by Hilbert in a completely different context: When he proved that Euclidean Geometry is consistent if Ordinary Arithmetic is consistent.

[3]I have called myself a Theoretical Physicicist only because I have a Ph.D in Theoretical Physics, but this was long ago. I had majored in Mathematics as an undergraduate and followed up a life-long study of mathematics both in work(research) and as a hobby.

theorist starts from Peano's Axioms and the axioms of logic with these as a basis the Geometrician/ Mathematican discovers new theorems which are essentially logical deductions from the Axioms. But a Theoretical Physicist starts from the study of a phenomena as observed in nature or as viewed in an experiment. He then tries to formulate "laws" which can explain this phenomena. Of course, there are limitations in both view points: The Mathematicain can never discover anything which lies beyond the reach of his axioms (this apect has been spectaculary demonstrated by Godel by his Incompleteness Theorem) and the Physicist can never discover any "Law" without having had the opportunity to view the phenomena or conduct an experiment.

So let me first summarize what has been done:

SUMMARY OF WHAT HAS BEEN DONE

1. It has been shown that for RH to be true, the condition given by Littlewoods theorem regarding the function $L(N)$ must hold.

In fact the real difficulty is that we are given only **"One instance"** of $\lambda(n)'s$. So if you (temorarily) fix N , then $L(N)$ is the distance traversed in an Nstep random walk. But this sequence is un-alterable and fixed whereas for an N step random walk we can have many random walks each of N steps, i.e we can have many instances. **The real challenge is: We are given only one instance of L(N) which is fixed,[4] then how can we compare with coin tosses when we are given only one sample?** The trick is that we are given only one instance of $\lambda(n)'s$ but **this is a sequence infinitely long!** We fully exploit this, because an infinite string (sequence) has infinite information: we can sample different lengths of the string (sequence) at different locations and compare with an equal length of coin tosses (e.g performing χ^2 fitting etc). **It is thus, by exploiting the infinite information available we could unravel the conundrum that was the RH.**

2. In Section 2 of Appendix VI of the Main paper, I have calculated consecutive $\lambda(n)'s$ forming a large sequence, denoted by $\Lambda[N_0, M]$, of length M of the form $\Lambda[N_0, M] \equiv \{\lambda(N_0),\ \lambda(N_0+1),\ \lambda(N_0+2), ...,\ \lambda(N_0+M-1)\}$.where $N_0$is some large integer.

It has been shown by actually computation and performing a χ^2 fit using the methods suggested by Donald Knuth,[5] that very large sequences containing consecutive values of $\lambda(n)'s$ very closely resemble and are infact "statistically indistinguishable" from "coin tosses of equal length. I have done very many computations (using Mathematica) and some of them have been presented in the Tables in Appendix VI. These are accurate and actual computations and the results are indisputable.

3(a) The sets of consecutive $\lambda's$ denoted as $S_+(N) = \Lambda(N+1, \sqrt{N})$ and $S_-(N) = \Lambda(N - \sqrt{N} + 1, \sqrt{N})$, (Na square integer) have the property of being "statistically indistinguishable" from coin tosses. The reason for this was demonstrably argued because each integer n occuring in the argument of $\lambda(n)$

[4] The $\lambda-$sequence was fixed ever since the time God made the integers!

[5] Knuth D.,(1968) 'Art of Computer Programming', vol 2, Chap 3. Addison Wesley

in one of the sets say $S_+(N)$ belongs to a different "Tower" [6](see Appendix VI).[7]

(b) In fact the union of all such sets i.e. $\cup_{k=1}^{\infty}(S_-(k^2) \cup S_+(k^2))$ covers the entire set of consecutive $\lambda(n)'s$ right upto infinity.

4. It has been shown that even though the lambdas satisfy the deterministic condition $\lambda(m.n) = \lambda(m).\lambda(n)$ the sequence of consecutive $\lambda(n)'s$ remain "statistically indistinguishable" from coin tosses for large n.

5. It was shown that for large n the $\lambda(n)'s$ satisfy the three criterion of (i) Equal probability (ii) unpredictability, (iii) independence and therefore using this "Principle of Statistical Analogy", one can deduce $L(N)$ has the asymptotic $C.\sqrt{N}$ behavior for large N, just like a random walk. Thus proving RH.

6. By using Khinchin and Kolmogrov's formulas of the iterated logarithm, a more accurate bound for the width of the ctitical line has been be obtained.

7. Additionally, an independent proof for showing the square root behaviour of $L(N)$ starting from Littlewood's Anzats was also obtained (Appendix V). Some theorems had alternative proofs.

8. Finally, the methods used not only proved RH but also gave reasons for the observed phenomena (from numerous numerical computations) of the consecutive $\lambda(n)'s$ behaving like a sequence of coin tosses, viz para 2 and paras 3(a) and 3(b) above.

CONCLUSION

A perusal of the above Summary shows how the Riemann Hypothesis is intimately connected with the factorization of integers as depicted in the $\lambda-$sequence and in the observed behaviour of the randomness of finite sets (e.g. $S_-(N)$, $S_+(N)$) of consecutive $\lambda(n)'s$, as demonstrated by numerous computations (Appendix VI)[8]. Further, the connection with RH arises from Littlewood's Theorem, which relates the growth of $L(N) = \sum_{n=1}^{N} \lambda(n)$ to the position of the critical line. Hence by investigationg the randomness of the $\lambda(n)'s$ and the randomness of coin tosses in particular by comparing the equations (1) and (3) with (2) and (4) respectively, and by making a through study of their statistical similarities and using the "Principle of Statistical Analogy" we finally arrived at the TRUTH of RH.

Regards

K. Eswaran

—

[6] The concept of "Towers" was introduced to understand, in an intutive way, the behavior of $L(N)$as depicted in the Figure 1 (see the Main Paper)

[7] These sets $S_+(N)$ and $S_+(N)$ can be very large, for example choosing $N = 10^{200}$the set $S_+(10^{200})$ has a length of a googol (10^{100}), i.e. it is a sequence containing googol consecutive $\lambda(n)'s$, guaranteed to be statistically indistinguishable from a googol number of consecutive coin tosses!

[8] These numerical computations and χ^2correlations are very real and are actually present and give very strong indications of "randomness" present in the $\lambda(n)'s$ which were actually computed by the factorization of integers n. *I strongly believe this phenomena has to be explained by the Pure Mathematicians and not brushed aside or put under the carpet or carelessly labelled as mere coincidence!* I believe my papers provides the ***raison d'etre*** for the existence of this phenomena.

YOUR PREVIOUS EMAIL of March 30 2020

Dear Professor Eswaran,

In my previous messages I tried to show you that your arguments are not very convincing. Your proof consists of the following steps:

1. You define predictability of a sequence in the following way:

The sequence a_n is predictable if for some integer M the term a_n (for large n) depends on M precedent terms.

This definition coincides with the definition of a recurrent sequence, used in several parts of mathematics.

2. You state that a sequence which is not predictable is independent, without defining the word "independent".

3. You recall a theorem from probability theory describing asymptotical behavior of the expected value of a series of independent random variables. I am not quite sure whether you quoted it correctly. I now it only for the case when the random variables attain the values 0 and 1. In the case when you have 1 and -1 it seems that the expected value would tend to zero. But I am not an expert in probability.

4. You state that this theorem implies that if a sequence is unpredictable (in the above sense) and attains the values 1 and -1 with the same frequency, then the sum of its first N values is asymptotic to$c\sqrt{N}$ with some c.

5. Applying this to the sequence $\lambda(n)$ one obtains the Riemann Hypothesis, as it is well-known that if for all large N one has

$$\sum_{n=0}^{N} \lambda(n) \leq c\sqrt{N}, \qquad (*)$$

then RH follows.

You do did not a proof the step 4, and actually the assertion stated there is incorrect, as one can easily construct a sequence a_n with $a_n = 1 or - 1$, which is unpredictable in the sense of 1, attains its values with the same frequency but does not satisfy the assertion (*). Example:

For$k = 1, 2, \ldots$ denote by I_k the interval$[(k-1) \cdot 10^4, k \cdot 10^4 - 1]$ and denote byA_k the set of all even integers in I_k to which$k^{3/4} \cdot 10^3$ odd integers from

I_k, taken at random, are added, and letB_k be the set of all remaining integers from I_k. Let A be the union of all sets A_k and B the union of all sets B_k, and define $A(x), B(x)$ to be the numbers o elements of the sets A, B in the interval $[1, x]$.

Define a function $f(n)$ by $f(n) = 1$ for $n \in A$ and $f(n) = -1$ for$n \in B$, and observe that because of the randomly added elements to$B$ the function$f$ satisfies your unpredictability condition.

One sees immediately that we have

$$\lim_{x\to\infty} A(x)/x = \lim_{x\to\infty} B(x)/x = 1/2,$$

and

$$\sum_{n\le k\cdot 10^4} f(n) = A(k\cdot 10^4) - B(k\cdot 10^4) = k^{3/4}\cdot 10^3 (**)$$

but according to your assertion 4 this sum should be bounded by $c\sqrt{k\cdot 10^4}$ $= c\sqrt{k}\cdot 10^2$ in contradiction to (**)

Therefore your proof is not correct.

Do not worry, your are not the first person who presented an incorrect proof of RH. At their list one finds some excellent mathematicians.

With every good wish

Wladyslaw Narkiewicz

END OF YOUR EMAIL 30th March 2020.

—

MY PREVIOUS EMAIL of March 17, 2020

Letter to Prof Wladyslaw Narkiewicz

Dear Professor Wladyislaw Narkiewicz,

I now define predictability of the $\lambda's$ follows.

DEFINITION: We say that $\lambda(n)$ is predictable if, there exists a finite integer M, such that for every $n > N_0$, the $\lambda(n)$, is derivable from its M previous values $\lambda(n-1), \lambda(n-2), \lambda(n-3), ...\lambda(n-M)$.

Note, the value of n is not explicitly known: only the values $\{\lambda(n-r), r = 1, 2, ..M\}$ are known.

We then assume that if $\lambda(n)$ is not predictable then it is independent.

Of course, if $\lambda(n)$, is derivable as above, then there exists some function f s.t :

$$\lambda(n) = f(\lambda(n-1), \lambda(n-2), \lambda(n-3), ...\lambda(n-M))$$

The justification for these definitions are given in Section 1 and Section 2.

SECTION 1:

Regarding your objection to the treatment of Eq(1) viz.

$$|L(N)| = \Sigma_{n=1}^{N} \lambda(n) \tag{5}$$

which by Littlewood's Theorem is supposed to satisfy the relationship:

$$|L(N) \leq C\, N^{a+\varepsilon} \tag{6}$$

for large N. In (2) I have used a in the exponent on the R.H.S. instead of 1/2, according to Littlewood if it so happens that $a = 1/2$. then RH is proved.

Comparing the above with the summation of a sequence "coin tosses" (or random walk) depicted by:

$$|S_N| = \Sigma_{j=1}^{N} X_j \tag{7}$$

where the random variable $X_j$takes on values $+1$ or -1, with equal probability and are independent of each other then, it is well known that for large N:

$$|S_N| \approx \sqrt{\frac{2}{\pi}}\, \sqrt{N} \tag{8}$$

Now, we have already shown that the $\lambda(n)$ has the probability of being $+1$ or -1 for very large n. In addition I have proved (in Appendix IV) that for large n, there is no way of knowing $\lambda(n+1)$ if we happen to know $\lambda(n)$.[9] This was sufficient to prove "independence" for large n. (Note if we know $\lambda(n)$ the next predictable value is $\lambda(2n) = -\lambda(n)$, and $2n$ is very far away from from n

[9] In fact, in Appendix IV, we have an alternative proof of 'independence' because we showed that the sequence of $\lambda's$ between $\lambda(n+1)$ and $\lambda(2n)$ are independent of each other; this length of this strip becomes infinite as n tends to infinity.

for large n.) Later, I used another criterion of "independence" surmising that if $\lambda(n+1)$ is derivable (predictable) from its M previous values, then the $\lambda(n)'s$ are predictable and hence not independent. But I showed that no fixed finite M exists[10] and hence we can consider the $\lambda's$ are independent for large n. (Therefore, this second criterion is more general because the first corresponds to M=1). We show in the second Section 2 that this definition of predictability is the correct one because by proving the $\lambda's$ are un-predictable (by this definition) and hence independent is able to explain the behavior ('phenomena') of the $\lambda's$ over large consecutive sections.

Hence we see that by using the same logic which derives (4) from (3) we can derive (2) from (1) and we will have the position of the critical vertical line at $a = 1/2$ thus proving R.H. However, in the paper I use a yet more rigorous analysis (see last paragraph of page 14 of the main paper) which was done in a study on independent random variables done by Khechin and Kolmogorov using iterative logarithms, their analysis gives the dependence of ε on N and how as N tends to infinity. From their result I have shown that the "width" of the critical line tends to zero, proving that all the nontrivial poles of F(s) and hence the zeros of $\zeta(s)$ lie on the critical line. QED

has

SECTION 2 : Phenomenological study of behavior of $\lambda(n)'s$

In Section2 of Appendix VI of the Main paper, I have calculated consecutive $\lambda(n)'s$ forming a large sequences of length $M$of the form $\Lambda[N_0, M] \equiv \{\lambda(N_0), \lambda(N_0+1), \lambda(N_0+2), ..., \lambda(N_0+M-1)\}$.where $N_0$is some large integer. And for each such case I did a χ^2 fit which compare sit with a Binomial sequence (coin tosses) of the same length M. as described by Knuth . In each case the value of $\chi^2 \leq 4.0$ showing that the $\lambda's$ in the sequence are indistinguishable from coin tosses. In the Tables given in the Appendix VI , I typically choose $N_0 = N$ (a perfect square and $M = \sqrt{N}$. E.G. the items in row 2 Table 1.4 show that the sequence of length $M = 100,000$ starting from $\lambda(10,000,000,001)$,......,to $\lambda(10,000,100,000)$ has a χ^2value equal to 1.15,thus it is statistically indistinguishable from a sequence of $100,000$ coin tosses or a random walk of $100,000$steps. This "phenomena' depicting the statistical behavior of the $\lambda's$ happens for all the sequences of ALL the entries in the Tables in Appendix VI. The reason for this phenomena is because each λ in the set $\Lambda(N, \sqrt{N})$ is sampled from a different 'Tower' and computing the summatory function $L(N, \sqrt{N})$ is like randomly picking a number from the next Tower which then may have value +1(H) or -1(T) with equal probability.[11]

Since all the sequences $\Lambda[N_0, M]$, of differing values of $N_0$and M, behave like coin tosses regardless of their starting value N_0, we can say that the statis-

[10] If no fixed finite M exists (i.e. M grows large with n or is infinite) it automatically means that the $\lambda(n+1)$ is all the more unpredictable from $\lambda(n)$ thus the $\lambda's$ are practically independent for large n.

[11] It is possible to argue similarly, and demonstrate that the sequence $\Lambda(N_0, M)$, for large N_0 and $M < 2N_0$will also statistically behave like coin tosses because each λ in the sequence will be drawn from a different Tower. and could be randomly +1(H) or -1(T).

tical behavior does not depend on the starting value N_0 ,further the statistics of $\Lambda[N_0 + k, M]$ (k being a small number) is the same as that of $\Lambda[N_0, M]$ we call this property as "Translational Invariance" which is certainly true if we are talking of the N_0^{th}coin toss $c(N_0)(\equiv X_{N_0})$; the coin tosses are all unpredictable and therefore independent. Now the numerous computed χ^2 values show that the sequences of the $\lambda's$ are statistically indistinguishable and also have the property of being statistically "Translational Invariant" for their corresponding sequences of coin tosses. Hence, to actually explain this phenomena of the statistical behavior we must show that the $\lambda's$ are also statistically "Translational Invariant", 'unpredictable' and therefore 'independent'. Hence we are *by tour-de-force* led to the following definitions:

DEFINITION: We say that $\lambda(n)$ is predictable if, there exists a finite integer M, such that for every $n > N_0$, the $\lambda(n)$, is derivable from its M previous values $\lambda(n-1), \lambda(n-2), \lambda(n-3), ...\lambda(n-M)$.

Note, the value of n is not explicitly known only the values $\{\lambda(n-r), r = 1, 2, ..M\}$ are known.

We then assume that if $\lambda(n)$ is not predictable then it is independent.

Of course, if $\lambda(n)$, is derivable as above then there exists some function f s.t :

$$\lambda(n) = f(\lambda(n-1), \lambda(n-2), \lambda(n-3), ...\lambda(n-M))$$

In the Main Paper we proved that such a function cannot exist otherwise it will make the λ-sequence cyclic, proving un-predictability and hence independence of the $\lambda's$.

From the above analysis we can conclude that as N tends to infinity the statistical behavior of the $\lambda's$ and the 'coin tosses' become very similar. Thus just as Eq.(4) follows from Eq. (3) for coin tosses, we are forced to conclude that Eq. (2) (with a=1/2) follows from Eq. (1) for the $\lambda's$.[12]

We have thus shown that the λ-series has a dominant statistical behavior of a random walk and therefore the summatory function $L(N)$ can be computed as is done in the paper using the iterative logarithm formula derived by Khechin and Kolmogorov, thus proving R.H.

It cannot be over emphasised that apart from proving RH, I have given reasons for the phenomena of large subsets sets of consecutive $\lambda's$ denoted as $S_+(N) = \Lambda(N+1, \sqrt{N})$ and $S_-(N) = \Lambda(N - \sqrt{N} + 1, \sqrt{N})$, (Na square integer) behaving like coin tosses, (see Eqs. (13) and (14), Sec. 4, Appendix VI of Main paper). This is important because by taking a collection of all perfect

[12]This is because the deterministic formula $\lambda(p.n) = \lambda(p).\lambda(n) = -\lambda(n)$, which can be used to predict the other values of the $\lambda's$ having known $\lambda(n)$ does not disturb the statistics of any large sequence, for example $\Lambda[N, \sqrt{N}]$ (N being square integer), this is because even if $\lambda(n)$ belongs to the sequence, even the very next predictable value $\lambda(2n)$ lies outside the range of the sequence and does not occur in $\Lambda[N, \sqrt{N}]$. The predictive power of the deterministic formula for $\lambda(m.n) = \lambda(m).\lambda(n)$, causes only a minor perturbation on the dominant statistical 'random walk' behavior of L(N), for very large N, and whose effect fades away at infinity.

In my project log in my Researchgate webpage, I have examined the behaviour of $\lambda's$ and its effect on L(N) and they have confirmed this assertion.

squares N, the sets $S_+(N)$and $S_-(N)$ contain all the $\lambda's$ upto infinity. Since, the Tables given in Appendix VI are actual computed values of the $\lambda's$ we have not only proved RH but have given an explanation of the behaviour of the $\lambda's$. Therefore, I believe that: Even if one is hard pressed to deny this proof of RH, then he/she is compelled to deny the very existence of this phenomena, but the latter cannot be denied because too many computations and χ^2comparisions have been confirmed. It is because of these reasons (and of course my own intution), I humbly believe that, what we have is a proof of RH!

In this reply I have confined myself to only your objections raised as points 1 and 2. I am leaving out point 3, as you have said a proof by von Mangoldt already exists. (I have given (in Ref 3, in the 1st Invitation Letter (email) dated 3rd Feb 2020, an alternative proof which relies on induction and not mapping, but I acknowledge that it has now become academic and redundant!).

To conclude, Professor Wladyslaw Narkiewicz, I want to once again express my thanks to you for sparing so much of your time reading my work. I do not know how to express my gratitude except by taking every word of yours seriously (thus demonstrating my profound respect for you) and try to make a complete clarification to the best of my ability. I will be most happy to clear any further doubts (if any). God Bless!

Regards
Kumar Eswaran
March 17, 2020
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END OF EMAIL of 17th March 2020

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"Władysław Narkiewicz" <Wladyslaw.Narkiewicz@math.uni.wroc.pl>

to: "Dr. Kumar Eswaran" <kumar.e@gmail.com>

date: Apr 3, 2020, 8:55 PM

subject: Re: Response to your Email of the 30th March Re: Riemann Hypothesis

mailed-by: math.uni.wroc.pl

Dear Professor Eswaran,

Thank you for your quick response. This thime I send you a short message with only one question. You find it in the attachment.

With best wishes

Wladyslaw Narkiewicz

SEE ATTACHMENT: From_Wladyslaw_April_3_2020_A-Short_Comment.pdf

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1. You assert that the theorem about random walks giving asymptotics $\sqrt{2/\pi}\sqrt{N}$ for the limit of $|E(S_N)|/N$ (your Equation (4)) implies for every $\varepsilon > 0$ the inequality

$$\sum_{n=1}^{N} \lambda(n) \leq c(\varepsilon) N^{1/2+\varepsilon}$$

with some positive $c(\varepsilon)$.

In your messages you gave also some vague indications of the proof, for example in your last message you wrote:

"*Therefore, any other sequence consisting of numbers* $+1$ *or* -1 *which satisfies the three above criteria (i), (ii) and (iii), will have its sum of* N *terms behave like (4)*", adding that there is no need of "*an elaborate machinery of probability spaces*".

Remember that there are two meanings of the word "probability". One, the common sense meaning, is applicable to events occurring in real life, and the other, used in mathematics, lives only in probability spaces. You **must** have a probability space Ω, a family of its subsets $X \subset \Omega$ and a function $p(X)$ with values in $[0, 1]$ satisfying certain conditions, otherwise one is unable to apply probabilistic theorems in other branches of mathematics. It seems that the problem of your proof lies in the fact that you disregard the difference of these two notions of probability.

If you really have a proof of your assertion, then I would like to be able to see it.

2. You pointed out that my example is wrong, but you did not observe that in the set A containing even numbers there are also bunches of odd numbers inserted at random, hence from $f(n) = 1$ the equality $f(n + 2) = 1$ follows not always.

Dr. Kumar Eswaran <kumar.e@gmail.com>

to: Władysław Narkiewicz <Wladyslaw.Narkiewicz@math.uni.wroc.pl>

bcc: "Dr. Kumar Eswaran" <drkumar_eswaran@yahoo.com>

date: Apr 5, 2020, 12:48 PM

subject: Re: Response to your Email of April 3rd

Dear Prof Wladyslaw Narkiewicz,

I thank you for your email of the 3rd April..

I have answered to my best of my ability the queries that you have raised.
Please see attached.

Also attached is an Extract of the bench mark paper by Prof S. Chanrasekar's (Nobel Laureate).I have made some annotations, for your convenience.

Thank You very much for your interest.

Regards
Kumar Eswaran

SEE ATTACHMENT: April_5_2020_Reply_Five_to_Wladyslaw.pdf

MY WRITE UP IN RESPONSE TO YOUR EMAIL of 3rd April 2020

5th April 2020

Dear Professor Wladyslaw Narkiewicz,

I thank you for the trouble you are taking over my papers.

1(a) I first address your first query in your para 1:

In this section I provide a gist of the requested proof, most of the details are in the Main paper and the Extended Abstract.

At the outset I wish to say that the summatory function $L(N)$:

$$|L(N)| = | \Sigma_{n=1}^{N} \lambda(n) | \tag{1}$$

and the summation of a sequence "coin tosses" (or random walk) depicted by:

$$|S_N| = | \Sigma_{j=1}^{N} X_j | \tag{2}$$

(where the random variable $X_j$takes on values $+1$ or -1, with equal probability), will behave statistically identically for large N, if ALL the following conditions are satisfied by $\lambda(n)$ viz. (i) Equal probabilities, (ii) independence (each "toss" is independent of the previous toss). The second condition effectively means "unpredictability' in Random Walk terms, (see Extended Abstract for details). **If all these conditions are satisfied then only we can conclude that,** for large N:

$$|L(N)\,| \leq C\,N^{1/2+\varepsilon} \tag{3}$$

Before answering your other queries, I wish to say that if there is a random variable X_j which takes on values $+1$ or -1, with equal probability and is independent i.e. they satisfy the criterion: (i) Equal probabilities and (ii) independence then, it can be proved that, for large N,

$$|S_N| \approx C\,\sqrt{N} \tag{4}$$

The proof of Eq.(4), following from the definition Eq.(2) and the criteria (i) and (ii) is a very well known result in the Statistical theory of random variables pertaining to coin tosses or 1-d random walks. See references in Footnote:[1]

[1] See e.g. (1) Chandrasekhar S., (1943)'Stochastic Problems in Physics and Astronomy',Rev.of Modern Phys. vol 15, no 1, pp1-87

(2) Khinchine, A. (1924) \Uber einen Satz der Wahrscheinlichkeitsrechnung", Fundamenta Mathematicae 6, pp. 9-20,

(3) Kolmogorov, A., (1929) \Uber das Gesetz des iterierten Logarithmus". Mathematische Annalen, 101: 126-135. (At the DigitalisierungsZentrum web site)). Also see: https:en.wikipedia.orgwikiLaw of the iterated logarithm.

NOTE: Ref.(2) and Ref.(3) gives a more accurate version of the $c.\sqrt{N}$ behavior, my Main Paper contains the details.

(4) One dimensional Random Walk https://mathworld.wolfram.com/RandomWalk1-Dimensional.html ▷

I am enclosing as a separate Attachment to this Email, an extract of the first 3 pages of Prof S. Chandrasekar's bench-mark paper where he deals with a 1-dimension Random Walk, pl. see pages 2 and 3 of this attachment the section The Simplest 1- dim. Random Walk Problem.

In his paper he shows for a Random walk: Eq.(4) follows from Eq.(2), notice that he uses only the two criteria (i) Equal probability and (ii) Independence to prove his result. I have Annotated and added a sticky notes on page 2 and 3, indicating the places where Chanrasekhar uses only the (i) and (ii) to derive Eq.(4) from Eq.(2).

Now comparing Eq.(1) and Eq.(2) we have the challenge that Eq.(1) represents a single instance of one sequence of $\lambda's$ which is given to us.[2] Where as Eq.(2) represents a generic case of a random walk (or coin tosses) and there can be many such random walks. The challenge is how, when we are given only one instance, can we extract its statistics? We overcame this challenge by exploiting the fact that though there is only one instance of Eq.(1) that is given to us, it is a sequence of infinite length. Therefore, we can examine very many sections of it and also use the fact that the $\lambda(n)'s$ have been obtained by the factorization of integers and are therfore governed by the laws of Arithmetic. By exploiting this fact we were able to prove (in the Main Paper) that the $\lambda-$sequence has (i) Equal probability and (ii) Independence.

But Chandrasekhar had shown that a sequence consisting of numbers $+1$ or -1, which satisfies the criteria (i) Equal probability and (ii) Independence, will have its sum of N terms behave like (4), and therefore (1) being an instance of a sequence which follows (i) and (ii) will also behave like (3) (because (4) and (3) are actually mean the same thing - differing only in notation). For convenience I call this as the "Principle of Statistical Analogy."[3] To repeat: Eq. (3) will follow from Eq. (1), because we have already proved (in the Main Paper) that the $\lambda(n)$ satisfies the criteria (i) and (ii). Thus RH necessarily follows from Littlewood's theorem. **QED.**

Additional Note: Littlewood's Theorem only requires that these criteria need only be satisfied by the λ-sequence for large N and we have proved that this is so.

1(b) Regarding your query on the definition of probability:

I define probability in the same manner as Prof. Chandrasekhar has done. Since a coin toss $X_j$can take values $[H,T]\equiv[-1,1]$ and this is similar to the values that $\lambda(n)$ can take viz. $[-1,1]$ there will be no contradiction.

All the definitions adopted by me has the virtue of being natural definitions and are simple and sufficient for my purposes: which was to show that the λ-sequence satisfy the criterion (i) and (ii).

[2]The $\lambda-$sequence was fixed ever since the time God made the integers!

[3]The principle of Analogy has been used by Hilbert in a completely different context: When he proved that Euclidean Geometry is consistent if Ordinary Arithmetic is consistent.

2. Coming to your second query labeled 2:

Your sequence certainly does not satisfy independence, because most of the time if n is odd $f(n) = f(n+2)$ and all of the time when n is even $f(n) = f(n+2)$. The sequence will $fail$ the test of independence. I simply cannot imagine that any person can experimentally toss coins for any length of time and get your result. If you still remain unconvinced on what I say, I suggest you look up the statistical tests of independence or ask a statistician.

The second fact that was overlooked (I forgot to mention this in my last email) when the "counter Example" was formulated, is that one must remember that the sequence $\{\lambda(n), n = 1, 2, 3, ..., \infty\}$ is not any arbitrary sequence: It is a sequence where for every integer n the value $\lambda(n)$ depends upon the number of prime factors of n and $\lambda(n) = +1\, or -1$ whether the number of prime factors are even of odd (multiplicities). Therefore the actual string $\{\lambda(n)\}$ is determined by the rules of arithmetic used for factorization a number using primes. For example: Consider the $\lambda-$sequence of 11 consecutive integers n (say) from 18 to 28,

$\{\lambda(18), \lambda(19), \lambda(20), \lambda(21), \lambda(22), \lambda(23), \lambda(24), \lambda(25), \lambda(26), \lambda(27), \lambda(28)\}$
$=\{\lambda(2.3.3), \lambda(19), \lambda(2.2.5), \lambda(3.7), \lambda(2.11), \lambda(23), \lambda(2.2.2.3), \lambda(5.5), \lambda(2.13),$
$\lambda(3.3.3), \lambda(2.2.7)\}$
$= \{-1, -1, -1, +1, +1, -1, +1, +1, +1, -1, -1\}$.

Your sequence $f(n)$ for (n even) satisfies $f(n) = f(n + 2)$ but the condition $\lambda(n) = \lambda(n + 2)$ can never happen in other words the rules of arithmetic and factorization cannot permit such a function $f(n)$ for any reasonable stretch of length, as proposed.

So one must remember that when one wishes to construct a **valid** "counter example", one must take care of (i) equal probabilities, (ii) independence (i.e no correlation) and (iii) also remember that the $\lambda-$sequence is determined by rules of factorization,[4] any arbitrary sequence simply will not do!

———

3. (a) Conclusion

Professor, out of regard to your high reputation of being a very good Number Theorist and because of the interest and trouble you have taken in reading my work I have considered every query of yours very seriously. I have as far as I can tell, have answered all your queries and I have actually nothing more to add. If, perchance, you have some difficulties with the statistical and other methods employed I suggest you could ask a statistician.

[4] The rules of factorization govern the value of every single $\lambda(n)$ but the rules of arithmetic work in such a manner that the sequence $\{\lambda(n), n = 1, 2, 3...., \infty$ becomes indistingusable from coin tosses as n appraches ∞.

3(b) Regarding Query

Now because you are a number theorist I make myself bold to ask one Query. Therefore, after perusing the following preamble I would be grateful if you could please attempt to answer my Query at the end of it.

Preamble to Query:

In Section 2 of Appendix VI of the Main paper, I have calculated consecutive $\lambda(n)'s$ forming a large sequence, denoted by $\Lambda[N_0, M]$, of length M of the form $\Lambda[N_0, M] \equiv \{\lambda(N_0),\ \lambda(N_0+1),\ \lambda(N_0+2), ...,\ \lambda(N_0+M-1)\}$.where $N_0$is some large integer.

It has been shown by actually computation and performing a χ^2 fit using the methods suggested by Donald Knuth,[5] that these very large sequences containing consecutive values of $\lambda(n)'s$ very closely resemble and are in fact "statistically indistinguishable" from "coin tosses of equal length. I have done very many computations (using Mathematica) and some of them have been presented in the Tables in Appendix VI,sec.3, e.g. Tables 1.3 and 1.4 page 27. These are accurate and actual computations and the numerical results are indisputable[6].

By very many numerical computations I have shown that the sets of consecutive $\lambda's$ denoted as $S_+(N) = \Lambda(N+1, \sqrt{N})$ and $S_-(N) = \Lambda(N-\sqrt{N}+1, \sqrt{N})$, (Na square integer) have the property of being "statistically indistinguishable" from coin tosses.[7] The reason for this was demonstrably argued because each integer n occurring in the argument of $\lambda(n)$ in one of the sets say $S_+(N)$ belongs to a different "Tower"

My Query:

These numerical computations and χ^2correlations are very real and are actually present and give very strong indications of "randomness" present in the $\lambda(n)'s$ which were actually computed by the factorization of integers n. *I strongly believe this phenomena has to be explained by the Pure Mathematicians and not brushed aside or put under the carpet or carelessly labeled as mere coincidence!*

Now I respecfully ask you, as a Number Theorist, how do you explain this phenomena of the random behavior of the $\lambda-$sequence?

Regards

Kumar Eswaran

---

[5] Knuth D.,(1968) 'Art of Computer Programming', vol 2, Chap 3. Addison Wesley

[6] I believe my papers provides the ***raison d'etre*** for the existence of this phenomena.

[7] Notice that if you choose $N = j^2$then the union of the two sets : $S_+(j^2+1, j) \cup S_-((j+1)^2 - j, j+1)$ is nothing but the sequence $\{\lambda(j^2+1),\ \lambda(j^2+2),\ \lambda(j^2+3), ...,\ \lambda((j+1)^2)\}$. That is they cover all the $\lambda's$ with arguments between two consecutive perfect squares, j^2 to $(j+1)^2$. Now if you choose $N = (j+1)^2$ you can cover the next region between the perfect squares $((j+1)^2+1)$ to $(j+2)^2$ and therefore you can capture all the regions between two consecutive perfect squares all the way up to infinity - basically covering all integers by the union of sets S_- and S_+right up to infinity.

"Władysław Narkiewicz" <Wladyslaw.Narkiewicz@math.uni.wroc.pl>

to: "Dr. Kumar Eswaran" <kumar.e@gmail.com>

date: Apr 14, 2020, 11:21 PM

subject: Re: Response to your Email of April 5

Dear Professor Eswaran,

I read very carefully your last message, but it did not convince me that you really have a proof of Riemann Hypothesis, as your arguments are of heuristical nature. I agree that the similarity of the considered sequence of values of the lambda-function with a random walk gives some reasons to believe in the truth of the conjecture. A similar idea appears already in the literature. In the attachment you will find a paper which perhaps will be of interest to you. It has been written by Good and Churchhouse published in the journal Mathematics of Computation (vol. 22, 1968, 857--861) a time ago with a similar heuristical approach to the Riemann Conjecture, based on the sequence of non-zero values of the Moebius function.

With best wishes
Wladyslaw Narkiewicz
SEE ATTACHMENT: Paper_sent_by_Wladyslaw_GoodChurchHouse.pdf

The Riemann Hypothesis and Pseudorandom Features of the Möbius Sequence

By I. J. Good* and R. F. Churchhouse

Abstract. A study of the cumulative sums of the Möbius function on the Atlas Computer of the Science Research Council has revealed certain statistical properties which lead the authors to make a number of conjectures. One of these is that any conjecture of the Mertens type, viz.

$$|M(N)| = \left|\sum_{n=1}^{N} \mu(n)\right| < k(\sqrt{N})$$

where k is any positive constant, is false, and indeed the authors conjecture that

$$\text{Lim sup } \{M(x)(x \log\log x)^{-1/2}\} = \sqrt{(12)}/\pi .$$

The Riemann zeta function is defined for $R(s) > 1$ by the series $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, and the definition is completed by analytic continuation. Riemann's hypothesis is that the "complex zeros" all occur where $R(s) = \frac{1}{2}$. It has now been verified for the first 2,000,000 complex zeros (Rosser and Schoenfeld, [8]), but this is not a very good reason for believing that the hypothesis is true. For in the theory of the zeta function, and in the closely allied theory of the distribution of prime numbers, the iterated logarithm $\log \log x$ is often involved in asymptotic formulae, and this function increases extremely slowly. The first zero off the line $R(s) = \frac{1}{2}$, if there is one, might have an imaginary part whose iterated logarithm is, say as large as 10, and, if so, it might never be practicable to find this zero by calculation. The plausibility of this argument is increased when we recall the refutation by Littlewood [5] of the conjecture that $\pi(x)$, the number of primes less than x, is always less than the logarithmic integral, $\mathrm{li}(x)$, a conjecture that is presumably true at least as far as $x = 10^9$ (see Ingham [4, p. 7]). It is possible that Littlewood had this kind of argument in mind when he said (Littlewood [6]), "In the spirit of this anthology (an anthology of partly baked ideas) I should also record my feeling that there is no imaginable *reason* why it (the Riemann hypothesis) should be true."

The aim of the present note is to suggest a "reason" for believing Riemann's hypothesis.

The Möbius function is defined by $\mu(n) = (-)^k$ if the positive integer n is the product of k different primes, $\mu(1) = 1$, and $\mu(n) = 0$ if n has any repeated factor. It is known (see, for example, Titchmarsh [9, p. 315]) that a necessary and sufficient condition for the truth of the Riemann hypothesis is that $M(x) = O(x^{1/2+\epsilon})$, for all $\epsilon > 0$, where $M(x) = \sum \mu(n)$ $(n \leqq x)$. The condition $M(x) = O(x^{1/2+\epsilon})$ would be true if the Möbius sequence $\{\mu(n)\}$ were a random sequence, taking the values -1, 0, and 1, with specified probabilities, those of -1 and 1 being equal.

Received May 1, 1967.

* Now at Virginia Polytechnic Institute, Blacksburg.

More generally, if we first select a subsequence from $\{\mu(n)\}$ by striking out all the terms for which $\mu(n) = 0$, and if this subsequence were 'equiprobably random,' i.e. if the value -1 and 1 each had (conditional) probability $\frac{1}{2}$, then the condition $M(x) = O(x^{1/2+\epsilon})$ would still be true. Of course a deterministic sequence can at best be 'pseudorandom' in the usual incompletely defined sense in which the term is used, and of course all our probability arguments are put forward in a purely heuristic spirit without any claim that they are mathematical proofs.

As a matter of fact there are *a priori* reasons, that is without looking at the numerical support, for believing the following conjecture:

Conjecture A. The sums of $\mu(n)$ in blocks of length N, where N is large, have asymptotically a normal distribution with mean zero and variance $6N/\pi^2$.

A priori *reasons for believing this conjecture*. Note first that $\mu(n) = 0$ when n is a multiple of 4, so there are zeros in the sequence $\{\mu(n)\}$ at regular intervals of 4, and similarly at regular intervals of 9 and so on. So, if we write ν_i for the number of values of n (in a block of length N) for which $\mu(n) = i$, we would expect ν_0 to be very close to its expected value

$$N - N(1 - 2^{-2})(1 - 3^{-2})(1 - 5^{-2}) \cdots = N(1 - 6\pi^{-2}) .$$

The numerical evidence for this statement is given in Tables 2 and 3 in the appendix.

Now if n is large and known to be square-free it is likely to have a fair number of factors, and therefore by the theory of the roulette wheel (with two sectors instead of 37 or 38) the probabilities that the number of factors is odd or even are nearly equal. Thus the probability that $\mu(n) = 1$(or -1) is near to $3\pi^{-2}$ and tends to this value when the range in which n is known to lie tends to infinity. Hence the expectation of $\mu(n)$ is 0.

The probability distribution of ν_1, conditional on a knowledge of ν_0, is binomial with mean $\frac{1}{2}(N - \nu_0)$ and variance $\frac{1}{4}(N - \nu_0)$. Allowing for the near-constancy of ν_0, the unconditional distribution of ν_1 would be expected to be binomial with mean $3N\pi^{-2}$ and variance $3N\pi^{-2}/2$. As a matter of fact the variance does not depend on the near-constancy of ν_0 since

$$\begin{aligned} \operatorname{var}(\nu_1 - \nu_{-1}) &= E(\nu_1 - \nu_{-1})^2 \\ &= E\{E[(\nu_1 - \nu_{-1})^2 | \nu_0]\} \\ &= E(N - \nu_0) = 6N\pi^{-2} . \end{aligned}$$

We have here assumed that, given ν_0, ν_1 has a 'heads-and-tails' binomial distribution. Its sample size is, of course, $N - \nu_0$.

This completes our *a priori* argument for believing Conjecture A, and even if Conjecture A is only approximately true it is so much stronger than the condition $M(x) = O(x^{1/2+\epsilon})$ that we feel its approximate truth would still support that condition.

We now describe the numerical test of Conjecture A, which was performed with $N = 1000$.

We computed $M(1000r + 1000) - M(1000r)$ (where we write $M(0) = 0$) for $r = 0(1)49{,}999$ on the Chilton Atlas (as a "background" job) but the values $r = 34{,}000(1)34{,}999$ were lost owing to a machine fault. Column (ii) of Table 1 gives the frequencies with which

$$M(1000r + 1000) - M(1000r)$$

lies in the ranges given by column (i). The "expected values" of these frequencies, based on Conjecture A, are listed in column (iii). For example,

$$7688 = \frac{49}{\sigma\sqrt{2\pi}} \int_{0.5}^{10.5} \exp\left(-\tfrac{1}{2}x^2/\sigma^2\right) dx\,,$$

where $\sigma^2 = 6000/\pi^2$. It will be seen that the fit is extremely good, in fact $\chi^2 = 24.2$ with 22 degrees of freedom. Thus Conjecture A is not merely to be expected *a priori*, by mathematical common sense, but it is well supported by the numerical data. As we said before, we believe therefore that there is a good "reason" for believing the Riemann hypothesis, apart from the calculation of the first 2,000,000 zeros.

TABLE 1.

Frequencies of the values of $\sum \mu(n)$ $(1000r < n \leqq 1000(r + 1))$, *for* 49000 *values of* r.

(i) *Range*	(ii) *Frequencies*	(iii) *Expectations*
−101 to −110	0	1.0
− 91 to −100	1	4.9
− 81 to − 90	16	20.6
− 71 to − 80	63	76.5
− 61 to − 70	248	240
− 51 to − 60	635	652
− 41 to − 50	1491	1465
− 31 to − 40	2797	2832
− 21 to − 30	4711	4645
− 11 to − 20	6604	6478
− 1 to − 10	7513	7688
0	793	794
1 to 10	7779	7688
11 to 20	6450	6478
21 to 30	4601	4645
31 to 40	2842	2832
41 to 50	1477	1465
51 to 60	651	652
61 to 70	216	240
71 to 80	88	76.5
81 to 90	16	20.6
91 to 100	7	4.9
101 to 110	1	1.0
	49000	49000.0

Although $\{\mu(n)\}$ is not a random sequence it is tempting to apply the law of the iterated logarithm (for example, Feller [1, p. 157]) to the subsequence obtained by deleting the values of n for which $\mu(n) = 0$. In this manner we generate a second conjecture, which, however, is less probable than Conjecture A and for which it is difficult to obtain numerical support. But it is of some interest to consider it.

Conjecture B.

$$\limsup \{M(x)(x \log\log x)^{-1/2}\} = \surd(12)/\pi .$$

Conjecture B contradicts Mertens's conjecture that $|M(x)| < x^{1/2}$, even in the extended form $|M(x)| < Cx^{1/2}$ for any constant C. When $C = \frac{1}{2}$ this modification of Mertens's conjecture was refuted numerically by Neubauer [7]: a breakdown occurred, for example at $x = 7.76 \times 10^9$. Also a conjecture of Pólya's, closely related to that of Mertens, was refuted by Haselgrove [3], who believed further that his method could be applied, with 1000 times as much calculation, to disprove Mertens's conjecture. In the light of this evidence, Mertens's conjecture is improbable, and our Conjecture B is somewhat supported by its inconsistency with it.

Appendix. Distribution of ν_0 for N = 1,000,000

In Table 2 we give the values of ν_0, i.e. the number of cases of $\mu(n) = 0$, in the first, second, . . ., 33rd block of length a million. We stopped at this point owing to the machine fault previously mentioned: a single supervisor fault caused the output for Table 1 to be lost at the 35th million and for Table 2 at the 34th million.

TABLE 2

Million	ν_0	*Million*	ν_0
1	392,074	18	392,088
2	392,049	19	392,039
3	392,104	20	392,037
4	392,037	21	392,072
5	392,103	22	392,084
6	392,076	23	392,096
7	392,053	24	392,047
8	392,101	25	392,096
9	392,061	26	392,071
10	392,051	27	392,071
11	392,073	28	392,065
12	392,078	29	392,079
13	392,073	30	392,065
14	392,095	31	392,083
15	392,083	32	392,077
16	392,093	33	392,076
17	392,057		

Using the 33 values of ν_0 given in Table 2, the estimated standard deviation is only 19.1. (The average number of zeros in each block of a million is very nearly 392,073.) This suggests that nearly always ν_0 is nearly constant in the sense that, for large N, the standard deviation of ν_0 is $o(\surd N)$, which for possible future reference we call Conjecture C. The total expected number of zeros of the sequence $\{\mu(n)\}$ in the first 33,000,000 is $33{,}000{,}000(1 - 6\pi^{-2}) = 12{,}938{,}405.6$ and the observed number is 12,938,407, an astonishingly close fit, better than we deserved.

The values of ν_0 for a few further values of N are shown in Table 3.

On the basis of this table we might even strengthen Conjecture C to (Conjecture D) 'the variance of ν_0 for large N is a constant.'

TABLE 3

N	ν_0	$E(\nu_0)$	$\nu - E(\nu_0)$
25,000,000	9,801,820	9,801,822.5	− 2.5
50,000,000	19,603,656	19,603,645	+11
75,000,000	29,405,440	29,405,467	−27
100,000,000	39,207,306	39,207,290	+16

The conjecture of the near constancy of ν_0 in each block of length N must be interpreted in an average ('probabilistic') sense, whether or not it is expressed in the form of Conjecture C. It would *not* be correct to assume that ν_0 is always close to its expected value; in fact it will sometimes though very rarely happen that $\nu_0 = N$. This will happen, for example, in the block $(M + 1, M + 2, \cdots, M + N)$ if simultaneously $M \equiv -1 \pmod 4$, $M \equiv -2 \pmod 9$, $M \equiv -3 \pmod{25}$, $\cdots, M \equiv -N \pmod{p_N^2}$, where p_N is the Nth prime. These congruences can be solved by Sun-Tsu's theorem (see, for example, Good [2, p. 759]). The value of M so obtained will be something like N^{2N}. Thus the Möbius sequence $\{\mu(n)\}$ contains arbitrarily long runs of zeros, but these long runs presumably occur extremely rarely.

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3. C. B. HASELGROVE, "A disproof of a conjecture of Pólya," *Mathematika*, v. 5, 1958, pp. 141–145. MR **21** #3391.
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8. J. B. ROSSER & L. SCHOENFELD, "The first two million zeros of the Riemann zeta-function are on the critical line," *Abstracts for the Conference of Mathematicians*, Moscow, 1966, 8. (Unpublished.)
9. E. C. TITCHMARSH, *The Theory of the Riemann Zeta-Function*, Clarendon Press, Oxford, 1951. MR **13**, 741.

Dr. Kumar
Eswaran <kumar.e@gmail.com
>

to:	Władysław Narkiewicz <Wladyslaw.Narkiewicz@math.uni.wroc.pl>
date:	Apr 16, 2020, 3:19 PM
subject:	My Response to your Email of April of the 14th April
mailed-by:	gmail.com

Dear Professor Wladyslaw Narkiewicz,

I thank you for email of the 14th April.

I have responded to it in the attachment, please see below.

I thank for initiating an illuminating discussion.

Keep Safe and God Bless
Regards
Kumar Eswaran

SEE ATTACHMENT: April-16_2020_Reply_Six_to_Wladyslaw.pdf
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MY RESPONSE TO YOUR EMAIL of 14th April 2020

Dear Professor Władysław Narkiewicz,

I thank you very much for your email of the 14th April.

I have to state as follows:

1. I was aware of Good and Churchhouse of 1968. They have done some numerical experiments supporting the speculation made by Denjoy in 1931 on the Mobius series, I have referred to Denjoy in my Main Paper see References, (Good and Church House have not acknowledged Denjoy).

Anyway, Good and Churchhouse are wrong for the following reasons:

(a) The Mobius series $\{\mu(n), n = 1, 2.3...\infty\}$ cannot be directly compared with a sequence of coin tosses because $\mu(n)$ can take one of three values namely $(1, -1, 0)$ depending on whether n has resp. even number of distinct prime factors or odd number of distinct prime factors or has a prime-square factor i.e. divisible by p^2 where p is a prime. Whereas a coin toss can take only two values viz: $+1$ (H) or -1 (T) Because of this, clear bounds cannot be obtained to estimate the behaviour of the sum $\Sigma_{n=1}^{N} \mu(n)$ for large N.

(b) More importantly, Good and Churchhouse **have not even tried to prove** by using, the laws of arithmetic (or number theory) (i) Equal probability and (ii) Independence.

Because of the above two counts: (a) and (b), their work does not count as a proof of RH. To be fair to them, they themselves acknowledge this, and do not claim that their paper has a mathematical proof.

However the same cannot be said of my work!

Reasons:

2. It has been shown by Khinchine & Kolmogorov that any sequence $\{X_j, j = 1, 2, ..\}$ made up of random variables X_j, which has the two properties (i) Equal probabilities of $X_j = +1 or -1$ and (ii) Independence will satisfy the relationship (for very large$N$):

$$| \, \Sigma_{j=1}^{N} X_j \, | \; \leq \; C \, N^{1/2 + \varepsilon} \qquad (A)$$

In my last Email I had enclosed, for your perusal, a Review paper by S. Chandrasekhar (Nobel Laureate), which derives the above from first principles starting from (i) and (ii).

3. I have proved that the sequence $\{\lambda(n), n = 1, 2, ...\infty\}$ made by terms belonging to the Liouville function $\lambda(n)$ which takes values $+1$ or -1, has precisely these two properties viz:(i) Equal probabilities of $\lambda(n) = +1 \; or \; -1$ and (ii) Independence (for large$N$): And therefore by the same logical argument must satisfy a similar relationship (for large N):

$$| \, \Sigma_{j=1}^{N} \lambda(j) \, | \; \leq \; C \, N^{1/2 + \varepsilon} \qquad (B)$$

The proof took most of my paper and involved the proof of may theorems for many of which I have given alternative proofs.

4. The way Eq. (A) follows from the properties (i) and (ii) for the sequence $\{X_j\}$ of $+1$'s and -1's, **implies** that any other sequence $\{\lambda(n)\}$ of of $+1$'s and -1's

which has been proved to satisfy the properties (i) and (ii) (for large N) **will necessarily obey Eq.(B) and thus RH follows.**[1]

This is a deduction made by using the principles of mathematical Logic. (Simply put: It means two mathematical entities which have similar properties will obey similar relationships). Therefore, with due respect, I wish to say that denying the logic, putting every thing done in the paper under the carpet i.e all the proofs, arguments and experimental evidence, and then dismissing them by using a single epithet: "heuristic", is equivalent to saying that mathematical logic is an unreliable tool and cannot be trusted if implemented in mathematical proofs. I simply cannot accept this stand because it completely destroys the very foundations upon which the edifice of mathematics and science are built.

In this email (and all my others) I have tried to expain that for my proofs, I had used only number theory, complex function theory and statistical concepts and mathematical logic throughout the length of my paper. And therefore I do not consider them "Heuristic" at all! But if you still think so, there is nothing more to be said.

With this I end.

I just want to thank you for initiating our illuminating discussion THANK YOU!

I hope the Covid19 pandemic is not affecting you and you are safe. From the news I found that nearly 8000 peope have been effected in Poland. Here in India it is around 12,000 but the worrying factor is that it is doubling every 4-5 days. I live in Hyderabad. DO TAKE CARE! Regards

Kumar Eswaran 16th April 2020

P.S. I have read many papers on RH, but very few of them have taken into account the randomness that appears in prime factorization. (The late Freeman Dyson and others first showed the connection of randomness of primes with the zeros of the zeta function). Most of such papers were involved in numerical computations and they did not investigate the reason for such random behaviour. Firstly, why should all the zeros be on a (critical) line? Secondly, when the few discovered the seeming evidence of randomness, they did not attempt, using the principles of arithmetic and number theory, to find the underlying reasons for such random behaviour.[2] It is primarily because of this, all of them have failed[3] to prove RH. I was driven to find the underlying reason for the appearence of randomness. I have been luck to have stumbled upon a solution, which truth to say, started off serendipitously.

[1]Note: From Littlewood's Theorem, Eq.(B) must hold for RH to be True. In fact, if one can invent a sequence which has the properties (i) and (ii) and possesss the properties of the sequence $\{\lambda(n)\}$, but **does not** satisfy Eq.(B), *then one would have actually disproved the Riemann Hypothesis!*

[2]Though some of them e.g. Polya, thought that the imaginary part of the zeros could be the eigenvalues of a Hermitian matrix and were driven to search for a quantum Hamiltonian whose eigenvalues will deliver the zeros, however such attempts were fruitless.

[3]In this particular case, I have found that reading old papers on RH, takes you far away from the ideas that actually would have lead you to a solution. It is better to read the present paper from a fresh stand point and not see through it as through the foggy windows of the past!

Władysław Narkiewicz" <Wladyslaw.Narkiewicz@math.uni.wroc.pl>

to: "Dr. Kumar Eswaran" <kumar.e@gmail.com>

date: Apr 17, 2020, 12:49 PM

subject: Re: My Response to your Email of April of the 14th April

mailed-by: math.uni.wroc.pl

Dear Professor Eswaran,

Although our views on your result are somewhat different I want you to know that I enjoyed our discussion which showed that the notion of a proof may have different interpretations. I want also to stress that the word "heuristic" has no negative meaning. A lot of work of really great mathematicians has been performed in a heuristic way. This applies not only to old times (Euler, Laplace, the Bernoullis, ...) but also to recent times.

Because of the pandemic I sit at home since three weeks, but this permits me to read all the books which I bought a time ago and did not have time to read them earlier. I am trying also to do some mathematics but my age limits my possibilities.

With every good wish

Wladyslaw Narkiewicz

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Dr. Kumar Eswaran <kumar.e@gmail.com>

to: Władysław Narkiewicz <Wladyslaw.Narkiewicz@math.uni.wroc.pl>

date: Apr 18, 2020, 12:10 PM

subject: Thank You For your clarification and some thoughts

mailed-by: gmail.com

18/4/2020

Dear Professor Wladyslaw Narkiewicz,

Thank you for your prompt email and gracious clarifications. Your kind words have greatly relieved me!

Yes, when I was working on the RH and beginning to realize how important the study of the growth and behavior of the lambda- sequence viz. {lambda(n), n=1,2,3…..N}, is to the problem of RH, I figured the following:

> (i) That some **better strategy,** than merely using Peano’s Axioms Dedekind-Peano's Axiom's ,

was needed for the proof of RH, because

> (ii) By solely using the Axioms to study the growth of the lambda-sequence up to infinity ,one would need knowledge of ALL the primes. (The axioms can only be used, in this case, to find primes and factors of each integer step by step).

All I can say is that **I have been lucky** in pursuing an alternative direction.

Your participation in our discussion has been very illuminating. Thank You.

And DO Take Care! And God Bless!

Regards

Kumar Eswaran.

Report on prof. K. Eswaran's proof of the Riemann hypotesis

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(Dated: 9 September 2020)

I. THE MAIN ARGUMENT

Professor Eswaran has proposed in references[1,2] a proof of the Riemann Hypothesis (RH) that will be summarized below. We first need some basic definitions. Let $\lambda(n)$ be the Liouville function that is a completely arithmetic multiplicative function equal to 1 (resp. -1) if the integer n is the product of an even (resp. odd) number of primes. Let $L(x)$ be the summatory function

$$L(x) = \sum_{n \leq x} \lambda(n) \, . \tag{1}$$

The RH is equivalent to the following statement

$$L(x) = O(x^{\frac{1}{2}+\epsilon}), \quad \forall \epsilon > 0 \ , \tag{2}$$

or alternatively

$$\lim_{x \to \infty} \frac{L(x)}{x^{\frac{1}{2}+\epsilon}} = 0, \quad \forall \epsilon > 0 \ . \tag{3}$$

This result is due to Landau and also appears in Littlewood works.

Prof. Eswaran's logic to prove the RH is based on demostrating eq.(2). The key argument consists in showing that the function $\lambda(n)$ behaves like a random walk, or a coin tossing game, in which case the sum of the outcomes up to N, would behave asymptotically as $L(N) \simeq C N^{1/2}$. If this were the case then certainly the RH would be proven !!

A general remark concerning Eswaran's approach is that the apparent random walk nature of the Liouville function is a well known fact in analytic number theory (see for example[3,4]). It is heuristic and, according to most expectations, unlikely to be true. The reason being that the prime numbers are deterministic objects, not random, as well as the Liouville and other arithmetic functions. These objects seem to behave randomly but they do not. Their apparent random nature has been very useful in the past to propose "conjectures" but it is impossible to use them to "prove" anything about the primes or related quantities.

An example of the heuristic use of the prime numbers is the twin prime conjecture of Hardy-Littlewood. The prime number theorem states that the number of primes below x behaves asymptotically as $x/\ln x$. The prime numbers look random with a probability density given by $1/\ln x$. However there are correlations between them that are described by the Hardy-Littlewood conjecture according to which the number of prime pairs $(p, p+2)$, below x behaves asymptotically as $C_2 x/\ln^2(x)$ with C_2 a constant. This conjecture is based on heuristic arguments but there is not a proof of it. There are many more examples of this type, which suggest that a proof of the RH based on random models is very unlikely. Similar methods have been applied to the Riemann zeros which seem to behave as the eigenvalues of hermitean random matrices. This idea has been used heuristically to conjecture the moments of the Riemann zeta function, but again there are not rigorous proofs except for a few cases.

These are general considerations to suspect why a proof of the RH based on random models is likely to be false, as has occurred in the past, but it does not exclude it. Below I present some arguments to show that, unfortutanely, Eswaran's approach fails to prove the RH.

The purported proof of the RH is based on the statement that the infinite set $\Lambda \equiv \{\lambda(1), \lambda(2), \lambda(3), \dots\}$ is an *instance* of an infinite random walk $\mathcal{R} = \{1, -1, -1, \dots, \}$ where every term behaves as a random variable $x_i = \pm 1$. For this to be the case the following conditions have to be satisfied (i) equal probabilities of $\lambda(n) = 1$ or $\lambda(n) = -1$, (ii) the λ-sequence has no cycle and (iii) unpredictability. Condition (i) follows from the Theorem in[1]

Theorem 3*: In the set of all positive integers, for every integer which has an even number of primes in its factorization there is another unique integer, (its twin), which has an odd number of primes in its factorization.*

This theorem is derived through an interesting partition of the positive integers into infinite sets $P_{m,p,u}$, called *towers*, whose elements, when ordered increasingly, have λ-values that alternate between $+1$ and -1. This representation is in turn used to provide a nice geometric interpretation of $L(N)$ as waves depicted in fig.1 of[1]. These partition of the natural numbers is also used to rewrite the Dirichlet series $F(s) = \sum_n \lambda(n)/n^s$ as a sum over the partitions $P_{m,p,u}$. But the main application of the towers is to provide a proof of theorem 3. Let me try to prove this theorem in a way that is inspired in these towers but is more simple and that will be used later on.

Theorem 3 is equivalent to the statement that all the positive integers, $\mathbb{Z}_+$, is the disjoint union of two sets $\mathbb{Z}_+^+$ and $\mathbb{Z}_+^-$, defined as

$$\mathbb{Z}_+^+ = \{n \in \mathbb{Z}_+ | \lambda(n) = +1\}, \qquad \mathbb{Z}_+^- = \{n \in \mathbb{Z}_+ | \lambda(n) = -1\} \tag{4}$$

that have the same cardinal. This means that there is a one-to-one correspondence between these two infinite sets. To proof this let us consider the pair of integers

$$\begin{aligned} n_0 &= 3^{e_3} 5^{e_5} \dots p_L^{e_L}, \qquad e_3, e_5 \dots, e_L \geq 0 \\ n_1 &= 2\, 3^{e_3} 5^{e_5} \dots p_L^{e_L} \end{aligned} \tag{5}$$

that satisfy $\lambda(n_1) = -\lambda(n_2)$. The values of e_a, determine to which set, $\mathbb{Z}_+^\pm$, the integers n_0 and n_1 belong to, but they are certainly different. The couple (n_0, n_1) form a *twin pair*, using the terminology of reference[1]. We can progress considering the integers

$$\begin{aligned} n_2 &= 2^2 3^{e_3} 5^{e_5} \dots p_L^{e_L}, \qquad e_3, e_5 \dots, e_L \geq 0 \\ n_3 &= 2^3\, 3^{e_3} 5^{e_5} \dots p_L^{e_L} \end{aligned} \tag{6}$$

that, by the same arguments as above, form a twin pair. The process can be interated at infinity and then all the integers can be organized in twin pairs such that finally $\mathbb{Z}_+ = \mathbb{Z}_+^+ \cup \mathbb{Z}_+^-$. The number 2 seems to play an special role in this construction but it does not. One could choose say 3, or any other prime number, and repeat the process obtaining the same result, namely $\mathbb{Z}_+ = \mathbb{Z}_+^+ \cup \mathbb{Z}_+^-$. Depending of the prime choosen one will obtain a different pairing which would amount to a permutation of the one-to-one maps between two infinite sets with the same cardinal.

Following reference[1], a conclusion of Theorem 3 is

Theorem 3B *: If n is an arbitrary positive integer,*

$$\text{Prob}[\lambda(n) = +1] = \text{Prob}[\lambda(n) = -1] = \frac{1}{2} \tag{7}$$

Strictly speaking this is not a theorem but a loose way to express theorem 3. One reason is that the definition of probability used in (7) is not given explicitly. If the sets $\mathbb{Z}_+^\pm$ where finite dimensional then the one-to-one correspondence between them would perhaps be expressed as in equation (7). But these sets are infinite dimensional.

A proper definition of probability, regarding the parity of $\lambda(n)$, can be given by considering the first N integers and counting the number of positive integers with the same value of $\lambda(n)$,

$$N_+ = \#\{n \leq N | \lambda(n) = +1\}, \qquad N_- = \#\{n \leq N | \lambda(n) = -1\}, \qquad N = N_+ + N_- \ . \tag{8}$$

The probabilities of finding $\lambda = \pm 1$ in the first N integers are given by

$$P_+(N) = \frac{N_+}{N}, \quad P_-(N) = \frac{N_-}{N} \tag{9}$$

whose difference is given by the summatory Liouville function $L(N)$,

$$P_+(N) - P_-(N) = \frac{N_+ - N_-}{N} = \frac{\sum_{n \leq N} \lambda(n)}{N} = \frac{\Lambda(N)}{N} \ . \tag{10}$$

The ratio in the last term of the r.h.s in the limit $N \to \infty$ was proven by Landau to vanish (doctoral thesis in 1899[3])

$$\lim_{N \to \infty} \frac{\Lambda(N)}{N} = 0 \ . \tag{11}$$

Landau showed that this result is equivalent to the Prime Number Theorem (PNT). In terms of the probabilities (9) we can write

$$\lim_{N\to\infty} P_+(N) = \lim_{N\to\infty} P_-(N) = \frac{1}{2} \ . \tag{12}$$

and interpret this result saying that *an integer has equal probability of having an odd number or an even number of distinct prime factors* (see page 8 of reference[3]). The latter statement is formulated in (7) as a theorem, but it is not. The theorem 3B is derived in[1] from theorem 3 and the properties of the towers, but the derivation is not convincing. It is true that within each tower the values of $\lambda(n)$ alternate, but going up in the list of integers, one jumps from one tower to another with no obvious pattern. It would be very interesting to use the tower construction to prove, not Theorem 3B that as explained earlier is not well formulated, but the PNT, namely equation (12). This is a much more easier target than the RH, but if achieved would represent an interesting result by itself.

The study of the properties of the Liouville function has been the focus on many works in the past that led to the development of new mathematical techniques. An interesting reference regarding this topic is[5] that among another results, proposes a probabilistic interpretation of the function $e^{-x/x}L(x)$ that nevertheless is quite different from a random walk.

II. CONCLUSION

In my opinion prof. Eswaran's works do not provide a proof of the RH. The methods employed to tackle this difficult problem do not even provide an alternative proof of the Prime Number Theorem. The idea of interpreting the Liouville sequence as random variables is attractive, although not original, but unfortunately has not been substantiated. The mystery of the primes remains untouched.

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[1] K. Eswaran, "The Final and Exhaustive Proof of the Riemann Hypothesis from First Principles", Research Gate, Vol. May 2018, https://www.researchgate.net/publication/325035649.

[2] K. Eswaran, "The Pathway to the Riemann Hypothesis", Preprint March 2019. DOI: 10.13140/RG.2.2.17950.59208. https://www.researchgate.net/publication/331889126.

[3] P. Borwein, S. Choi, B. Rooney and A. Weirathmueller (eds). "The Riemann Hypothesis: A Resource for the Afficionado and Virtuoso Alike", CMS Books in Mathematics, Springer (2008).

[4] I. Good and R. Churchhouse, "The Riemann Hypothesis and Pseudorandom Features of the Möbius Sequence", Mathematics of Computation. 22, 857 - 864 (1968).

[5] Peter Humphries, "The distribution of weighted sums of the Liouville function and Pólya's conjecture", Journal of Number Theory 133 (2013) 545-582.

Your Open Review of my paper on RH

from: **Dr. Kumar Eswaran** <kumar.e@gmail.com>

to: German Sierra <german.sierra@uam.es>

bcc: "P. Narasimha Reddy" <nrriemann@sreenidhi.edu.in>

date: Oct 6, 2020, 9:55 PM

subject: Your Open Review of my paper on RH

mailed-by: gmail.com

Dear Professor German Sierra,

I thank you for the trouble you took in participating in the Open Review of my paper on RH.

I am attaching my reply to your comments.
Kindly read the document, I will be happy to receive your response.
Since I have referred to the Lectures given by Vinayak (on my proof) in my reply, I have attached his lectures for your convenience.

Kind Regards
Kumar Eswaran
Professor
Enclosures:
1) My Reply to your comments
2) Vinayak's Lectures on my proof.

Reply to Prof German Sierra Comments on RH

Dear Professor Sierra,

I thank you for comments on my paper that you have submitted to the Panel of Scientists.

I wish to reply as follows:

1 Regarding your Comments

I believe you have a fundamental misunderstanding regarding the role of "randomness" in my paper. You, rightly, point to several examples of previous work where randomness was assumed and, again rightly, say that those were at best conjectures which left their conclusions heuristic, which you say is the most that can be said of my proposed proof. However, all those cases you cited were mostly regarding the conjectured *randomness of primes* and its implications. But all these matters play no role in my proof.

In fact I make no attempt to prove the "randomness of primes" (a conjecture which is very difficult to prove and, luckily for me, is not required for the proof of RH). What I use in my proof is, the randomness of the sequence of $\lambda(n)$ (which appears as the n^{th} term in the Liouville series), in the sense that the λ-sequence (series) resembles a random walk, in that (a) $\lambda(n) = -1\, or + 1$ are equally probable and (b) $\lambda(n)$ is not dependent on previous values of the λ's upto any *finite distance*, as the length $N$of the sequence tends to ∞. This statistical resemblence to a random walk for large N, persists even though some individual members of the λ's are related by the deterministic relation $\lambda(mn) = \lambda(m).\lambda(n)$.

These considerations, I then show is enough to prove RH, using the Khinchin and Kolmogorov's Law of the Iterated Logarithm. Both (a) and (b) can be proved in more than one way and I have done so in my paper. Actually, as I have shown, the fact that λ's are deterministic and obey the deterministic relation $\lambda(m.n) = \lambda(m).\lambda(n)$, and therefore not random in the classical sense, does not matter[1] at all for large N. If (a) and (b) can be proved then RH is proved, as you yourself have agreed (see paragraph following Eq.(3) in your review).

In your comments[2] you have mostly confined yourself to only a part of the actual arguments that lead to the result I called Equal Probability. *It was **NOT** my intention to say that Equal Probabilities is enough to prove RH. But this is precisely what you seem to have thought because there is no reference by you to*

[1]For example, given $\lambda(n)$ for some n, the formula $\lambda(m.n) = \lambda(m).\lambda(n)$,can determine the next predictable value $\lambda(2.n) = \lambda(2).\lambda(n) = -\lambda(n)$, but since we consider very long sequences (length $N \to \infty$) for a large n, say $n = 10^{100}$, the integer $2n$ will be at a distance of 10^{100} from n making such a prediction statistically insignificant!

[2]I am well aware of the works you cited in your references, namely Borwein etal. your Ref [3], Good and Churchhouse, your Ref [4] and Peter Humpries, your Ref [5] . While the first two deal mostly with the Mobius function and the third more with Polya's conjecture. (In footnote 4, I briefly state the difficulties in using the Mobius function). None of them actually make a detailed study of of the growth of L(N) $= \sum_{n=1}^{N} \lambda(n)$ as $N \to \infty$ nor do they make a rigorous comparision with a random walk (or coin tosses).

the rest of the paper. I feel that this is an unfortunate misunderstanding, which I wish to resolve, but I do, very much, need your kind patience and forebearance. This is what was done in the rest of the paper. The purpose of this note is to write out the rest of the Arguments.

Going back to Equal Probability. In the main paper I used a mapping and pairing technique to obtain the result. I acknowledge that there are various other prescriptions of mapping to prove the result. I also gave an Alternative proof which does not use mapping but just pure induction (see my paper, Ref [2], in Research gate). My intention was to tackle RH from **first principles,** this is the reason for my giving the two proofs.

In these proofs, while I focused on mapping, an alternate and equivalent interpretation could be that I partition the number system into infinite subsets such that each natural number (except 1) can be uniquely assigned to only one such subset. Each of these subsets have members whose λ values alternate between $+1$ and -1 and therefore it is evident that its members have an equal probability of having their λ values either $+1$ or-1. As any randomly chosen natural number n will belong to one such subset, it will have an equal probability of having $\lambda(n)$ equal to $+1$ or -1,which is the Equal Probability result.

Yes, I too was very well aware that if one uses the Prime Number Theorem and the fact that $\zeta(s)$ has no zero on the vertical line $Re(s) = 1$ and also Littlewoods theorem (stated in my paper) it is possible to prove Equal Probabilities see slide 238...in Ref [6] (below)Vinayak's lecture[3]. I did not include this because as the title of my paper has the phrase "from first principles".

However, as you have rightly pointed out that the Equal Probability result is obtainable from the Prime Number Theorem. If so, you have accepted that (a) is proved. What remains to prove is (b) that, the $\lambda(n)$ is independent of previous $\lambda's$ (within a finite distance) in a long sequence of length $N \to \infty$. Therefore they would closely resemble the statistical behavior of a ramdom walk (coin toss).

The independence of $\lambda(n)$, for large arguments n,i.e. condition (b), is shown in two different ways. In the first (see Sec 11.3 Appendix IV of Ref.[1]) while it is acknowledged that the $\lambda's$ are linked through the multiplicative relationship $\lambda(mn) = \lambda(m).\lambda(n)$ etc., it is shown that the numbers linked in this manner move increasingly further from each other so that this distance tends to infinity as $n \to \infty$. Thus in the sequence $\{\lambda(n)\}$ each $\lambda(n)$ is independent of preceding $\lambda's$ which lie within a finite distance from it as $n \to \infty$.

In the second proof of Independence (see para 2(a), p 21, in Ref[1]), it is shown that any functional relationship which binds $\lambda(n)$ to its previous $\lambda's$ (which lie within a distance L, arbitrary but fixed) and which is valid for all n,would make the sequence $\lambda(n)$ periodic after some large $n > n_0$. It is then proved that the sequence $\lambda(n)$ cannot have such a cyclic behavior, because this would imply, from Littlewood's theorem, that there are no zeros of the zeta function within the critical strip - which is not true.

[3] Vinayak Eswaran (my brother), had taken the trouble to write out 7 lectures on my proof of RH, see Ref [6]. The lectures are well worth reading because of their clarity, lack of jargon and assumes only the pre-reqisite of an under-graduate level of mathematical foundation.

With the properties (a) and (b) proved, a direct application of the Khinchine-Kolmogorov Law of the Iterated Logarithm shows that the RH is TRUE and that the width of the critical line is zero!

In the next section onwards I outline the main steps of the proof, which will be a useful to read the paper. (I also strongly recommend Vinayak's Lectures Ref [6] in the Reference Section (below), the lectures provide a detailed, and sometimes an alternative, version of my proof).

In the last I have added an experimental Verification which is not a part of the proof. But the extensive calculations verify the properties of the lambda's over very large sequences and hence lend support and credence to our methods.

2 Gist of The Proof (See Ref. 3 for Details)

The proof proceeds in essentially 4 basic steps.

STEP 1: We choose[4] an analytic function, $F(s) = \zeta(2s)/\zeta(2s)$, whose poles exactly correspond to the non-trivial zeros of the zeta function $\zeta(s)$. $F(s)$ is analytic in the region $Re(s) > 1$and is given by:

$$F(s) = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} \tag{1}$$

STEP 2: An analytical continuation of $F(s)$ to the left of the vertical $Re(s) = 1$, using Littlewood's Theorem determines that the asymptotic behavior of the summatory function $L(N)$:

$$L(N) = \sum_{n=1}^{N} \lambda(n) \tag{2}$$

as $N \to \infty$ plays a crucial role in determining the analyticity of $F(s)$ and the position of the poles of F(s) and thereby the zeros of $\zeta(s)$ in the critical region $0<$ Re(s) <1. Littlewood's theorem states that the asymptotic behaviour of $L(N)$ for large N, determines the analyticity of F(s), and if the behaviour is such that

$$\mid L(N) \mid \equiv \mid \sum_{n=1}^{N} \lambda(n) \ \ \mid < \ C\, N^{a+\epsilon} \qquad (for\ large\ N\,) \tag{3}$$

(where $(1/2 \leq a < 1\,)$, and ϵ is a small positive number), F(s) will be analytic in the region $(a < Re(s)$. This is a very crucial result as far as RH is concerned

[4]It may be mentioned here that for his study Littlewood had chosen the function $1/\zeta(s)$ which had lead to the μ-sequence. A difficulty with using this Mobius function is that $\mu(n)$ can take values$-1, 0, or + 1$, whereas $\lambda(n)$ takes values of $-1\, or + 1$, like a coin toss. Because of this it is easier to compare the λ-sequence with coin tosses rather than the μ-sequence. And because of this very reason I chose to study the function $F(s) = \zeta(2s)/\zeta(s)$, which leads to the λ-sequence.

because if one can determine that actually $a = 1/2$ in (3) then the Riemann Hypothesis is proved.

STEP 3: In this step we logically put forth the argument: *that the very necessity that (3) must be satisfied for the Riemann Hypothesis to be true, imposes very severe resrtictions on the behaviour of the sequence of the Liouville functions:* $\{\lambda(1), \lambda(2), \lambda(3), ...\}$. These restrictions (conditions) are given later in this section.

The $\lambda(n)$ is defined as: $\lambda(1) = 1$and for $n > 1$: $\lambda(n) = (-1)^{\Omega(n)}$ and is determined by factorizing n and finding $\Omega(n)$, the number of prime factors of n (multiplicities included). We already know $\lambda(n)$ is fully determined by factorizing n and is an arithmetic function namely: $\lambda(m.n) = \lambda(m).\lambda(n)$, for all integers m, n.

Now for RH to be true one must have $a = 1/2$ in Eq.(3), the first N terms (N large) of the λ sequence must therefore sum up as:

$$| \lambda(1) + \lambda(2) + \lambda(3) + ... + \lambda(N) | \simeq C \, . \, N^{1/2} \tag{4}$$

The above equation brings to mind a similar relationship satisfied by another sequence $c(n) = \pm 1$, which corresponds to the n^{th} step of a One-dimensional random walk! (This $c(n)$ can be simulated by coin tosses, if we replace Heads by $+1$ and Tails by -1; so a N-step random walk can be thought as a coin toss experiment where a coin is tossed N times.) It is well known that for such a random-walker's sequence the sum indicates the distance travelled from the starting position in N steps and satisfies the relationship:

$$| c(1) + c(2) + c(3) + ... + c(N) | \simeq C \, . \, N^{1/2} \tag{5}$$

The well known result of Equation (5), (see S.Chandrasekar), is derived by using the assumption that the random walker behaves in such a manner that:

(i) Each step is of the same size but can be either in the positive direction or negative direction i.e the nth step $c(n)$ can be $+1$ or -1, with Equal Probability.

(ii) The sequence of steps cannot be periodic, that is the pattern of steps cannot form a repetitive pattern (there is no cycle) [5].

(iii) Knowing the n^{th} step the $(n+1)^{th}$ cannot be predicted. That is, knowing c(n), c(n+1) cannot be determined (they are independent).

The above assumptions are enough to derive Eq(6). This has been shown by many researchers (e.g. See S. Chandrasekar, refered in Ref[1])

2.1 The Argument:

Comparing (4) and (5) leads us to deduce some inevitable conclusions:

Eq.(4) must be satisfied by the $\lambda(n)$ sequence *if the Riemann Hypothesis is TRUE*, this is the conclusion that we deduce from Littlewoods Theorem, with the proviso that Eq. (4) needs be satisfied only for large N (this being

[5]The requirement that there are no cycles was not necessary for Chandrasekar, but it is necessary for the proof of (iii) i.e. independence.

the condition of Littlewood's theorem). Now, there are many Random walks possible, for instance: 100 random walkers can each of them, take N steps and each of these random walkers will be at anapproximate distance of distance $C\,.\,N^{1/2}$from the starting point. Each of these 100 sequences can be thought of as *100 different instances of a random walk of N steps each.*

If we wish to compare (5) with (4) there are several conceptual issues: (α) The sequence in (4) is a deterministic sequence and (β) we have only one sequence. We get over this latter issue by considering the single sequence as *one instance* of a hypothetical random walk of N steps. And even though the $\lambda(n)$'s are deterministic (an aspect we temporarily ignore) we could investigate *this one instance* and argue (or hypothesize) that when N is large, the following rules could be obeyed:

Properties of the $\lambda-$sequence

(a) Given an arbitrary large n chosen at random, there is Equal probability of $\lambda(n)$being either $+1$ or-1.

(b) The $\lambda-$sequence cannot be periodic, that is the $\lambda(n)$ cannot form a repetitive pattern (no cycle)

(c) Knowing the value of $\lambda(n)$ it is not possible to predict $\lambda(n+1)$. Unpredictability (independence).

Note the rules (a),(b)and (c) are similar to (i),(ii) and (iii) and therefore: *If by using the number theoretical (arithmetical) properties of the integers, the primes and the factorization process, it is somehow possible to prove that the $\lambda(n)$ satisy the rules (a), (b) and (c) then just as (5) is satisfied by every instance of a random walk, Eq(4) will be satisfied for our one particular instance of our $\lambda-$sequence and thus RH will be proved! Taking this as a cue we proceed. NOTE: According to Littlewood's Theorem: It is only necessary that the $\lambda-$ sequence satisy the rules for very large lengths of the sequence and large arguments of λ. It is because of this relaxation provided by Littlewoods theorem that even though the $\lambda-$sequence is deterministic, but its behaviour still very closely approximates to the statistical behaviour of a sequence of random walks (or coin tosses) over large N.*

Hence the next step is to prove the properties for large values of N i.e. when N tends to infinity. (It will also become clear later that the deterministic nature of the $\lambda(n)'s$, does not significantly disturb the above statistical properties.[6])

However, we are actually now at the crossroads: we have to prove that the $\lambda-$sequence possesses the above properties (a),(b) and (c) for large sequence lengths and large arguments N. Property (a) has already been proven, it is quite possible that by using the artithmetic properties of the $\lambda(n)$ that (b) can be proved (as has been done in the Appendix III of the Main Paper) but it is in the proving of (c) that the real difficulty lies. This is because all mathematical proofs in Arithmetic relies heavily on the Axioms of Peano (P.A), but P.A. does not come to our aid for certain hard problems e.g to prove (or disprove) that the advent of primes are random. Luckily we don't have to decide upon this last surmise! But, we do have to decide upon the problem of resolving

[6]See Foot note 1, on page 1.

what is "independence" or "unpredictability" of a sequence. So we define that a sequence $\{a\} \equiv \{a(1), a(2), ...a(n), ...\}$ as unpredictable, for large values of its arguments, if when given the value of $a(N)$ where N is large and the M previous values where $M << N$ (and M finite), viz.$\{a(N-M), a(N-M+1), ..., a(N-1), a(N)\}$ then it is not possible to predict $a(N+1)$. It can be easily seen that if a sequence $\{a\}$ has this property then since, it is not possible to predict the value of $a(N+1)$ knowing $a(N)$ and its M previous values, we can assert that $a(N)$ and $a(N+1)$ are independent. It will be proved that by this definition the $\lambda-$*sequence* $\equiv \{\lambda(1), \lambda(2), ..., \lambda(N), ...\}$the components of $\lambda(N)$ and $\lambda(N+1)$, are independent for large values of N. Using this knowledge we can proceed.

STEP 4: Proof of the Properties of the $\lambda-$sequence. In this step several theorems are proved using the number theoretical (arithmetical) properties of integers, primes and the unique factorization of integers to establish the properties (a), (b) and (c) of the $\lambda-$sequence as listed in the previous paragraphs. These proofs are fairly straight forward and are from first principles:

We have already seen that Property (a) On Equal Probabilities, is proved in Theorems 2 and 3 in Section 5.2, in the Main Paper Ref [1] and that the concept of "towers" is used in the proofs. An alternative proof[7] by constuction of all prime products and induction is also given in Ref [2]. A third proof, which follows from Littlewood's theorem but assumes the fact that there is no zero with Re(s)=1, (proved in the Prime Number Theorem) can also be derived as has been discussed in the foregoing and also by you (but is not given in the paper).

Property (b) On no cycles, is proved in Appendix III, Ref [1]

Property(c) On unpredictability (independence) is proved in Appendix IV, Ref[1]. An alternative proof of this also given: See para 5(a), in page 2, of Ref [3].

An alternative arithmetical proof of the asymptotic relation $\mid L(N) \mid \simeq c = C.N^{1/2}$. is given in Appendix V, Ref [1].[8]

[7]This alternative proof of equal probabilities is given in Ref [2] and is done by explict construction of integers by products of sets which are powers of a given prime. Each of these sets are seen to be comprised of members of ascending magnitudes and alternating $\lambda-$values, in perfect analogy with the ascending and alternating odd/even sequences of the natural numbers. Thus the member n of each set has the exact probability of 1/2 (as $n \to \infty$) of its $\lambda-$value being $+1$ or -1 as the natural numbers have of being odd or even. As every natural number can be placed uniquely in one such subset, it is seen that a randomly chosen natural number will have a probabilty of 1/2 of having its $\lambda-$value equal to $+1$ or -1(as $n \to \infty$).

[8]In a separate arithmetical study Ref [4], it was discovered that for very large N, smaller primes contribute more (than the larger primes) to the calculation of $\lambda(n)$'s which occur in the summatory expression for $L(N)$. Specifically, if one chooses an integer K such that $K << N$ then the primes p which are s.t $p < N/K$, occur much more often in the calculation of each term in $L(N)$ than the large primes q which are s.t. $N/K < q < N$. This situation permits us to deduce, interestingly, that if we allow both $K \, and \, N \to \infty$ in such a manner that the ratio N/K is a fixed number, then we must have: $Pr(\lambda(n) = 1 \mid n < N) = 1/2 - \frac{C_K}{logN}$ and $Pr(\lambda(n) = -1 \mid n < N) = 1/2 + \frac{C_K}{logN}$ where $C_K$is a small fluctuating number which tends to zero as $K \to \infty$; thus once again confirming that the $L(N)$ behaves like a random walk for very large N.

We have therefore showed that Eq. (3) is satisfied by the $\lambda-$sequence. We will now get an expression for the "width" of the critical line and show that this width vanishes in the limit of large N.

We have established the fact that the $\lambda-$sequence behaves like a coin tosses (or a random walk) and this *entitles* us to use Khinchine -Kolmogorov's law of the Iterated Logarithm, adapted to the present context, is: Let $\{\lambda_n\}$ be independent, identically distributed random variables with means zero and unit variances. Let $S_N = \lambda_1 + \lambda_2 + ... + \lambda_N$. The limit superior (upper bound) of S_N almost surely (a.s.) satisfies

$Lim\ Sup\ \frac{S_N}{\sqrt{2N\ loglogN}} \ = 1 \quad as\ N \to \infty$

Now, from Theorem 4 we have written that if we consider the $\lambda's$ as "coin tosses" one can write $\mid L(N) \mid = \mid \lambda(1) + \lambda(2) + ... + \lambda(N) \mid \leq \ C_0\, N^{\frac{1}{2}+d_N}$ (as $N \to \infty$) (since we are interested in only the behaviour for large N we henceforth ignore the constants). Comparing this expression with the one above we see that one can write $N^{\frac{1}{2}+d_N} \sim \sqrt{N\ loglogN}$ thus yielding an expression for $d_N = \frac{logloglogN}{2logN}$. We see that $d_N \to 0$ as $N \to \infty$, this satisfies the equivalent statement of (3) viz. $\frac{L(N)}{N^{\frac{1}{2}+\epsilon}} = 0$, for any chosen $\epsilon > 0$ (Ref. Eq.(20) in Ref.[1]) . Further $d_\infty \equiv (Lim_{N\to\infty}\, d_N)$ is the half-width of the critical line. Since this is zero, we conclude that all the non- trivial zeros of the zeta function must lie strictly on the critical line.

Thereby, the Riemann Hypothesis is proved.

3 Experimental verification

In the last Appendix VI, of the Main Paper, Ref[1], numerical experiments (using Mathematica) are described and there it is shown that large sequence of lambdas behave like a random walk (or equivalently like coin tosses).

I have calculated consecutive $\lambda(n)'s$ forming a large sequence, denoted by $\Lambda[N_0, M]$, of length M of the form $\Lambda[N_0, M] \equiv \{\lambda(N_0),\ \lambda(N_0+1),\ \lambda(N_0+2), ...,\ \lambda(N_0+M-1)\}$.where $N_0$is some large integer.

It has been shown by actually computation and performing a χ^2 fit using the methods suggested by Donald Knuth,[9] that these very large sequences containing consecutive values of $\lambda(n)'s$ very closely resemble and are in fact "statistically indistinguishable" from a Binomial distribution ("coin tosses) of equal length. I have done very many computations (using Mathematica) and some of them have been presented in the Tables in Appendix VI,sec.3, e.g. Tables 1.3 and 1.4 page 27; also see "End Note" on the last page of this document. These are accurate and actual computations and the numerical results are indisputable[10].

By very many numerical computations I have shown that the sets of consecutive $\lambda's$ denoted as $S_+(N) = \Lambda(N+1, \sqrt{N})$ and $S_-(N) = \Lambda(N-\sqrt{N}+1, \sqrt{N})$, (Na square integer) have the property of being "statistically indistinguishable"

[9] Knuth D.,(1968) 'Art of Computer Programming', vol 2, Chap 3. Addison Wesley

[10] I believe my papers provides the ***raison d'etre*** for the existence of this phenomena.

from coin tosses.[11]

These sequences called $S_-(N)$ and $S_+(N)$ exist (N being a perfect square) and behave like random sequences (coin tosses) *and the concatination of such sets of $S_-(N)$ and $S_+(N)$ cover all of $\lambda(n)$ for all integers n upto infinity.* This shows that the entire $\{\lambda(n)\}$ sequence is made up of an infinite series of subsequences of type $S_-(N)$ *and* $S_+(N)$ each of which statistically behave like coin tosses! The Tables 1.4 cited above, provide ample proof of this. The purpose of this section is just to demonstrate that the, predictions of the theorems proved in the Main Paper, have been numerically verified extensively. The verifications[12] have been done by doing a χ^2 fit of a $\lambda-$sequence with a Binomial distribution (coin tosses). In every case it has been shown that for large N the $\lambda-$sequence is indistinguishable from a random walk (sequence of coin tosses).

These numerical computations and χ^2correlations are very real and are actually present and give very strong indications of "randomness" present in the $\lambda(n)'s$ which were actually computed by the factorization of integers n. *I strongly believe this phenomena has to be explained by the Pure Mathematicians and not brushed aside or put under the carpet or carelessly labeled as mere coincidence!*

I wish to emphasize, that I have not only explained this phenomena but also showed how it connects with the proof of the Riemann Hypothesis.

4 Conclusion

In this write up I have shown that the reason for the RH to be true lies with the fact that the λ- sequence behaves statistically like coin tosses. It was shown that a sequence $c(n)$ of coin tosses or a sequence of a random walk, exhibits the square root behavoir of Eq(4), was deduced by Khinchine-Kolmogorov, Chandrasekar and others, from the assumption of two criteria (i) Equal Probabilities (ii) Independence. By using the properties of arithmetic and from the use of mathematical deductions we could prove many theorems (and many have alternative proofs) to show that the λ- sequence for large values of its arguments also satiisfy (i) and (ii). And therfore they satisfy Littlewood 's condition for

[11]The reason for this was demonstrably argued because each integer n occurring in the argument of $\lambda(n)$ in one of the sets say $S_+(N)$ belongs to a different "Tower".

Notice that if you choose $N = j^2$then the union of the two sets :

$S_+(j^2 + 1, j) \cup S_-((j + 1)^2 - j, j + 1)$ is nothing but the sequence

$\{\lambda(j^2+1), \lambda(j^2+2), \lambda(j^2+3), ..., \lambda((j+1)^2)\}$. That is they cover all the $\lambda's$ with arguments between two consecutive perfect squares, j^2 to $(j + 1)^2$. Now if you choose $N = (j + 1)^2$ you can cover the next region between the perfect squares $((j + 1)^2 + 1)$ to $(j + 2)^2$ and therefore you can capture all the regions between two consecutive perfect squares by concatinating such sets all the way up to infinity - basically covering all integers by the union of sets S_- and S_+right up to infinity.

[12]*In fact I have proved (in Appendix VI of the Main Paper) that if you do a χ^2 fit of a sequence of $\lambda's$ of length N with a sequence of coin tosses of equal length (using Knuth's method) then $\sum_{n=1}^{N} \lambda(m+n) = \chi\sqrt{N}$ for N large. If necessary, one may consult the extensive numerical calculations done at the end of the last Appendix involving $\lambda-$sequences as long as 100,000 and argumets of λ as large as 10, 000,000,000!*

RH to be true. We also made extensive numerical computations involving the $\lambda-$sequence; all these support the various theorems we have proved. Since, we have started from first principles and used only the arithmetic properties of numbers and mathematical logic to prove the theorems, in my opinion, this leaves hardly any doubt as to the truth of RH.

My appeal to you: The RH is the greatest open problem in Mathematics. I believe that I have solved it comprehensively. I request you to look at the proof again (starting with the reading of Vinayak's lectures which can be done quickly), as I found your review had missed many aspects which I believe, are very crucial to the proof, (I beg your pardon for saying so!). This onerous task, which I am requesting you to kindly take up, will not only do justice to my work but also to the great problem it addresses. If you have any further queries or doubts, it will be my pleasure to answer them.

In the End, as in all matters of consequence, Truth will prevail.

Thank You. And kind Regards

K. Eswaran

5 References

[1] The final and Exhaustive proof of the Remann Hypothesis...

[2]A Simple Proof That Even and Odd Numbers of Prime Factors Occur with Equal Probabilities in the Factor-ization of Integers

[3]A Quick Reading Guide to the Proof of the Riemann Hypothesis

[4] The effect of of the non-random-walk behavior of the Liouville Series L(N) by the first finite number of terms.

I enclose below the slides of the Invited Lecture that I delivered at the Government Arts & Science College Kumbakonam on March 1st 2019. (This was followed by another (slightly shorter) Lecture delivered in the Ramnujan Centre of Sastra University on the evening of the same day).

[5] Invited Lecture On the Riemann Hypothesis by K.Eswaran

[6] Vinayak Eswaran: Seven Lectures of Kumar Eswaran's Proof on RH

K. Eswaran/Professor
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5th October 2020

6 END NOTES *-These are just end notes which -provide some more information.*

Just for curiosity I tested the behaviour of very long sequences of lambda and compared them with coin tosses by χ^2 fitting.See the Appendix VI of my Main Paper Ref[1], there are many more examples in the form of Tables.

Here we define the summation: SUM $\equiv \sum_{n=K}^{K+L} \lambda(n) = \chi \sqrt{L}$ (see Eq(9), in page 25 of Ref[1]).

Everywhere the χ^2 fits get better and better as the the length of the sequence and the size of the integer increases. Knuth speculated that a value of around 4 or 5 to make the sequence indistinguishable from a sequence of coin tosses or a random-walk. However, Littlewood's criterion is far less strict, it is sufficient that as N tends to infinity: $\sum_{n=K}^{K+N} \lambda(n) = C \sqrt{N}$, where C can be any finite value.

In the tests below the lambda sequence passes the test in every case. I have taken very Long sequences, for Example III, I have considered 100,000 consecutive integers starting from $K = 25 \times 10^{24} + 1$

EXAMPLE I: A sequence of Length L of consecutive lambdas starting from $\lambda(25000001)$ to $\lambda(25005000)$ of 5000 λ values for 5000 consecutive integers,starting from $25,000,001$ ie $L = 5000$. We use Mathematica commands in our computations as shown below.

Plus[LiouvilleLambda[Range[25000001, 25005000]]]

= {1, 1, 1, 1, 1, 1, 1, -1, -1, -1, 1, -1, -1, 1, 1, 1, -1, -1, 1, 1, \ 1, -1, -1, 1, 1, -1, 1, 1, -1, 1, 1, -1, -1, 1, 1, -1, 1, -1, 1, -1, \ -1, -1, 1, -1, -1, -1, -1, 1, -1, -1, 1, -1, 1, -1, 1, -1, 1, 1, 1, \TRUNCATED-- \ 1, -1, 1, 1, 1, 1, 1, 1, -1}

SUM= -42, χ^2= [SUM * SUM/ L] = [42*42/5000}= 0.3528

EXAMPLE II

Starting Integer $K = 25 \times 10^{24} + 1; L = 10,000$

Plus[LiouvilleLambda[Range[25000000000000000000000001, 25000000000000000000010000]]]

SUM= 88, χ^2 = 88*88/10000 = 0.7744

EXAMPLE III:

Starting Integer $K = 25 \times 10^{24} + 1; L = 100,000$

Plus[LiouvilleLambda[Range[25000000000000000000000001, 25000000000000000000100000]]]

SUM= 238, χ^2 = 238*238/ 100000 = 0.56644

EXAMPLE IV:

Starting Integer Z = $10^{30} + 1$; L=1000;

Plus[LiouvilleLambda[Range[Z, Z + 999]]]

SUM= -20, χ^2 = 20*20/1000 =0.4

EXAMPLE VI:

Starting Integer Z = $10^{30} + 1$; L=10,000;

Plus[LiouvilleLambda[Range[Z, Z + 9999]]]

SUM= 54 χ^2 = 54*54/10000 = 0.2916

Part B: K Eswaran All Main Papers on RH

The Final and Exhaustive Proof of the Riemann Hypothesis from First Principles

Preprint · May 2018

DOI: 10.13140/RG.2.2.35243.95528

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Eswaran, K.. (MAY 2018) “The Final and Exhaustive Proof of the Riemann Hypothesis from First Principles,”
Research Gate, Vol. MAY 2018, Article ID Version - FINAL
doi of Reseachgate/PREPRINT/Version - FINAL

The Final and Exhaustive Proof of the Riemann Hypothesis from First Principles

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ABSTRACT: As is well-known, the celebrated Riemann Hypothesis (RH) is the prediction that all the non-trivial zeros of the zeta function $\zeta(s)$ lie on a vertical line in the complex s-plane at $Re(s) = 1/2$. Very many efforts to prove this statement have been directed to investigating the analytic properties of the zeta function, however all these efforts have not been able to substantially improve on Riemann's initial discovery: that all the non trivial zeros lie in verical strip of unit width whose centre is the critical line. The efforts have been rendered difficult because of a lack of a suitable functional representation (formula) for $\zeta(s)$ (or $1/\zeta(s)$) , which is valid and analytic over all regions of the Argand plane; these difficulties are further complicated by the presence of prime numbers in the very definition of the zeta function and the lack of predictability in the behaviour of prime numbers which makes the analysis intractable. In this paper we make our first headway by looking at the analyticity of the function $F(s) = \zeta(2s)/\zeta(s)$ that has poles in exactly those positions where $\zeta(s)$ has a non trivial zero. Further, the trivial zeros of the zeta function, which occur at the negative even integers, conveniently cancel out in $F(s)$ and do not appear as poles of the latter (however there is an isolated pole of $F(s)$, viz. $s = 1/2$, which is actually a pole of $\zeta(2s)$ but this will not worry us because it is on the critical line). So the task of proving the RH is some what 'simplified' because all we have to show is: All the poles of $F(s)$ occur on the critical line, which then is the main aim of this paper. We then investigate the Dirichlet series that obtains from the function F(s) and employ novel methods of summing the series by casting it as an infinite number of sums over sub-series. In this procedure, which heavily invokes the prime factorization theorem, each sub-series has the property that it oscillates in a predictable fashion, rendering the analytic properties of the function $F(s)$ determinable. With the methods developed in the paper many theorems are proved, for example we prove: that for every integer with an even number of primes in its factorization, there is another integer that has an odd number of primes (multiplicity counted) in its factorization; by this demonstration, and by the proof of several other theorems, a similarity between the factorization sequence involving (Liouville's multiplicative functions) and a sequence of coin tosses is mathematically established. Consequently, by placing this similarity on a firm foundation, one is then empowered to demonstrate, that Littlewood's(1912) sufficiency condition involving Liouville's summatory function, $L(N)$, is satisfied. It is thus proved that the function $F(s)$ is analytic over the two half-planes $Re(s) > 1/2$ and $Re(s) < 1/2$, clearly revealing that all the nontrivial zeros of the Riemann zeta function are placed on the critical line $Re(s) = 1/2$.

Extended Abstract

The paper approaches* the RH in the following way:

(1) This proof of the Riemann Hypothesis (Riemann 1859) crucially depends on showing that the function $F(s) \equiv \zeta(2s)/\zeta(s)$, has poles only on the critical line $s = 1/2 + iy$, which translates to having the non-trivial zeros of the $\zeta(s)$ function on the self-same critical line. It can be easily verified that all the non-trivial zeros of $\zeta(s)$ appear as poles in $F(s)$, and all the trivial zeros cancel and so do not appear as poles in $F(s)^\dagger$. It can also be proved, from symmetry considerations, that both the numerator and denominator of $F(s)$ cannot vanish at the same point. Hence, to prove the RH, all we need to show is that all the poles of $F(s)$ occur on the critical line.

(2) A method applied by Littlewood (1912, see Edwards (1974) pp 260) to obtain an equivalent statement of RH involving the $1/\zeta(s)$ function is applied here to $F(s)$ to obtain a previously-known equivalent statement of

Received 9 MAY 2018
Communicated by XXX

*__This paper is the Final Version of the proof on RH, the earlier versions bear the title 'The Dirichlet Series for the Liouville Function and the Riemann Hypothesis' Sept 2016 to Oct 2017) are listed in Ref. [8] K. Eswaran__
†except that the latter has an extra pole on the critical line

uncertain provenance. Littlewood's method lies in analytically continuing the Dirichlet series for $F(s)$, strictly valid for $Re(s) > 1$, to the region $Re(s) \leq 1$. The analyticity of $F(s)$ turns out to be crucially dependent on the boundedness of $L(N)N^{-s}$ as $N \to \infty$, where $L(N)$ is the summatory Liouville function. That is, if $L(N) \sim N^a$ $(a > 0)$ asymptotically as $N \to \infty$, then the $F(s)$ can be analytically continued only on the right of the vertical line drawn at $Re(s) = a$. In other words, the singularities of F(s) will lie on or to the left of $Re(s) = a$. Further, since the non-trivial zeros of the Riemann zeta function $\zeta(s)$ exactly correspond to the poles of F(s), and are known to be symmetrically placed about the $Re(s) = 1/2$ line, this automatically implies that the non-trivial zeros of the zeta function will all be within the region $1 - a \leq Re(s) \leq a$. If $a \to 1/2$ from the right, the zeros will lie on the critical line and RH will be true. This also means that $L(N) \sim N^{\frac{1}{2}}$ as $N \to \infty$, thereby yielding the RH-equivalent statement (see Borwein et al 2006, p. 48):

$$lim_{N\to\infty} \frac{L(N)}{N^{\frac{1}{2}+\epsilon}} = 0 \quad \text{for}\, \epsilon > 0$$

Here $L(N) = \sum_1^N \lambda(n)$, and $\lambda(n)$ is the Liouville function that is +1 or -1 depending on whether n has an *even* or *odd* number of prime factors (multiplicity included).

(3) The equivalent statement above requires, if the RH is true, that $|L(N)| \sim N^{\frac{1}{2}}$ as $N \to \infty$ ($a < 1/2$ is impossible as the $Re(s) = 1/2$ line is known to have numerous zeros of the the zeta function). This expression is strongly suggestive of $X(N)$, the distance travelled in N unit steps in a standard *random walk*, which can be represented as:

$$X(N) = \sum_{n=1}^{N} c(n)$$

where the $c(n)$'s are "coin-tosses", i.e., independent random numbers with an equal probability of being either +1 or -1. It is well-known that the expected value of $|X(N)|$, for large N, is

$$lim_{N\to\infty} E(|X(N)|) = C_0 N^{1/2}$$

Therefore, if it could be shown that the $L(N)$ series is a random-walk, and that $|L(N)| \sim N^{\frac{1}{2}}$ as $N \to \infty$, the RH would be proved. This is the approach taken here. So we have to prove that the $\lambda(n)$'s in the $L(N)$ series are essentially "coin-tosses", for large n.

To show that the λ's behave as coin tosses, we have to show that (i) their probabilities of being either +1 of -1 are equal, and further (ii) that the λ's appearing in the natural sequence, $n = 1, 2, 3, ...$, are independent of each other.

(4) *Equal Probabilities*: A crucial advance in this line of attack is the discovery of a method of factorizing every integer and placing it in an exclusive subset, where it and its other members in the same subset form an increasing sequence of natural numbers that alternately have odd and even numbers of prime factors. Such subsets are called 'Towers' in the Paper. It is shown that every natural number, other than 1, has a unique place in a unique Tower of ordered countably infinite members, and each such member represents a unique natural number. The alternating odd and even factorization of the members of the Towers ensures that each Tower is partitioned into equal proportions of members with λ's of -1 and +1. As the entire natural number system is incorporated within the Towers system, the natural numbers also have equal proportions with odd and even numbers of prime factors, respectively with λ's of -1 and +1. In other words, concerning the probability that a random natural number n has a λ of either value, we conclude: $Prob[\lambda(n) = +1] = Prob[\lambda(n) = -1] = 1/2$ [Theorem 3B][‡].

(5) *Non-periodicity and Independence:* We then show that the sequence of λ's in $L(N)$ can never be cyclic, just as a sequence of coin tosses can never be cyclic. This done in Appendix III and follows directly from Littlewood's method. Quasi-cyclicity or any other pattern of λ's in the $L(N)$ series that would keep $L(N)$ bounded as $N \to \infty$ are also excluded. Specifically, non-cyclicity would preclude any dependence of the type

$$\lambda_n = f(\lambda_{n-1}, \lambda_{n-2}, \lambda_{n-3}, ..., \lambda_{n-M})$$

[‡]Theorems 3 and 3B, that argue for equal proportions and hence equal probabilities of the the λ's over the natural number system, were later given a new and different proof in K. Eswaran, (April 2018)

for finite M. So, essentially, the independence of λ's *in the natural sequence* is proved [although *within the Towers*, they have a perfectly predictable and deterministic relation]. Here we adopt the notation: λ_n for $\lambda(n)$.

(6) *Arithmetic Independence:* The independence of the λ's in the $L(N)$ series mentioned above followed from Littlewood's method, i.e., Analysis. In Appendix IV, a purely arithmetic approach is taken. It is shown that from merely these two rules: $\lambda(p) = -1$ for a prime p, and $\lambda(pq) = \lambda(p) \times \lambda(q)$, where q is any integer, the entire sequence of λ's for $n = 1, 2, 3, ...$, can be obtained, in a manner reminiscent of the construction of the natural number system by multiplication. It is then argued that for any two integers $n > m$, $\lambda(n)$ is dependent on $\lambda(m)$, if the latter is required to find the former, and independent if not. It is then shown that, as $n \to \infty$, any finite sequential strip of λ's will be independent of each other, thus essentially making them equivalent to coin-tosses.

(7) With Theorem 3B and Appendices III and IV, we have proved that the $L(N)$ series is a random walk of infinite length. We then invoke (towards the end of Section 5) Khinchin and Kolmogorov's law of the iterated logarithm to show that the maximum deviation, from the one-half power-law expectation, in the exponent of $|X(N)|$ for any individual random walk tends monotonically to zero as $N \to \infty$. So, in fact, $|L(N)| \sim N^{\frac{1}{2}}$ as $N \to \infty$. Therefore, for any chosen $\epsilon > 0$, the equivalent statement for the validity of RH will be satisfied and the Riemann Hypothesis is proved.

Interestingly, the aforementioned maximum deviation from the one-half power-law expectation in the random walk of $L(N)$ is exactly the half-width of the critical strip around $Re(s) = 1/2$ that contains all the zeros of the of the zeta function. As that deviation approaches zero as $N \to \infty$, that width is also zero, ensuring that all the non-trivial zeros of the zeta function lie on $Re(s) = 1/2$.

(8) In Appendix V, starting from Littlewood's *ansatz*, that $|L(N)| \sim N^a$, for $N \to \infty$, we argue that the statistics of the λ's must become "self-similar" over large consecutive sequences of λ's. It is shown that, if we choose two sets $S_-(N)$ and $S_+(N)$ of consecutive integers, each of them containing k integers,[§] then the λ's defined over these sets $S_-(N)$ and $S_+(N)$ are statistically similar to each other. This statistical similarity is shown to hold for all large k i.e for all $N = k^2$, which implies the statistical behavior of λ's are independent of the length k of $S_-(N)$ and $S_+(N)$ for large but arbitrary $N = k^2$. This principle yields us the value of $a = \frac{1}{2}$, which again would satisfy the equivalent statement of the RH. This 'physicist's proof' of the RH, is separate from the argument in the main paper, and may be treated as an interesting addendum to it.

(9) Finally, in Appendix VI, we show the sequence of λ's is statistically indistinguishable from coin tosses (using the χ^2 statistical test) over many sets of consecutive integers (as was demonstrated in Appendix V). Further it is also shown that the sequence λ's is indistinguishable from coin tosses over the entire range of numbers from $n = 1$ to 176 *trillion.* This verification has been done by actual numerical computation over large sets of integers (which are below 176 trillion). While this is merely a 'verification', not a 'proof', this empirical result follows directly from our proof of the Riemann Hypothesis, and affirms its sound basis.

Interestingly, it is also shown in Appendix VI that a connection exist between $L(N)$ and χ^2 when compared to a sequence of coin tosses. From the relation Eq.(9), p 24, one can conclude that to satisfy Littlewood's condition, only the first and second moments of the distributions of lambda and coin toss sequences need be similar.

Because of the extensive computations and calculations made, which are backed by theory, Appendix VI can be thought of as an experimental physicists' verification of the 'Law of Riemann'.

1 Introduction

This paper investigates the behaviour of the Liouville function, (ref. Apostol (1998), which is related to Riemann's zeta function, $\zeta(s)$, defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{1.1}$$

where n is a positive integer and s is a complex number, with the series being convergent for $Re(s) > 1$. This function has zeros (referred to as the trivial zeros) at the negative even integers $-2, -4, \ldots$. It has been

[§]The set $S_-(N)$ contains k consecutive integers ending with the integer $N = k^2$, and $S_+(N)$ contains the next k consecutive numbers, see Appendix V for details and definitions shown¶ that there are an infinite number of zeros on the line at $Re(s) = 1/2$. Riemann's Hypothesis (R.H.) claims that these are all the nontrivial zeros of the zeta function. The R.H. has eluded proof to date, and this paper demonstrates that it is resolvable by tackling the Liouville function's Dirichlet series generated by $F(s) \equiv \zeta(2s)/\zeta(s)$, which is readily rendered in the form

$$F(s) = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}, \tag{1.2}$$

where $\lambda(n)$ is the Liouville function defined by $\lambda(n) = (-1)^{\Omega(n)}$, with $\Omega(n)$ being the total number of prime numbers in the factorization of n, including the multiplicity of the primes. We would also need the summatory function $L(N)$, which is defined as the partial sum up to N terms of the following series:

$$L(N) = \sum_{n=1}^{N} \lambda(n) \tag{1.2b}$$

Since the function $F(s)$ will exhibit poles at the zeros of $\zeta(s)$, we seek to identify where $\zeta(s)$ can have zeros by examining the region over which $F(s)$ is analytic. By demonstrating that a sufficient condition, derived by Littlewood (1912),(in Edwards(1974)), for the R.H. to be true is indeed satisfied, we show that all the nontrivial zeros of the zeta function occur on the 'critical line' $Re(s) = 1/2$.

Briefly, our method consists in judiciously partitioning the set of positive integers (except 1) into infinite subsets and couching the infinite sum in (1.2) into sums over these subsets with each resulting sub-series being uniformly convergent. This method of considering a slowly converging series as a sum of many sub-series was previously used by the author in problems where Neumann series were involved Eswaran (1990)).

In this paper we break up the sum of the Liouville function into sums over many sub-series whose behaviour is predictable. It so turns out that one prime number p (and its powers) which is associated with a particular sub-series controls the behaviour of that sub-series.

Each sub-series is in the form of rectangular functions (waves) of unit amplitude but ever increasing periodicity and widths - we call these 'harmonics' - so that every prime number is thus associated with such harmonic rectangular functions which then play a role in contributing to the value of $L(N)$. It so turns out that if N goes from N to $N + 1$, the new value of $L(N + 1)$ depends solely on the factorization of $N + 1$, and the particular harmonic that contributes to the change in $L(N)$ is completely determined by this factorization. Since prime factorizations of numbers are uncorrelated, we deduce that the statistical distribution of $L(N)$ when N is large is like that of the cumulative sum of N coin tosses, (a head contributing +1 and a tail contributing -1), and thus logically lead to the final conclusion of this paper.

We found a new method of factoring every integer and placing it in an exclusive subset, where it and its other members form an increasing sequence which in turn factorize alternately into odd and even factors; this method exploited the inherent symmetries of the problem and was very useful in the present context. Once this symmetry was recognized, we saw that it was natural to invoke it in the manner in which the sum in (1.2) was performed. We may view the sum as one over subsets of series that exhibit convergence even outside the domain of the half-plane $Re(s) > 1$. We were rewarded, for following the procedure pursued in this paper, with the revelation that the Liouville function (and therefore the zeta function) is controlled by innumerable rectangular harmonic functions whose form and content are now precisely known and each of which is associated with a prime number and all prime numbers play their due role. And in fact all harmonic functions associated with prime numbers below or equal to a particular value N determine $L(N)$. The underlying symmetry being alluded to here, remained hidden because the summation in (1.2) is written in the usual manner, setting $n = 1, 2, 3, ...$ in sequence.

From the next section onwards the paper follows the plan enunciated in the Extended Abstract and indicated by the the steps (1) to (9) detailed therein.

2 Partitioning the Positive Integers into Sets

The Liouville function $\lambda(n)$ is defined over the set of positive integers n as $\lambda(n) = (-1)^{\Omega(n)}$, where $\Omega(n)$ is the number of prime factors of n, multiplicities included. Thus $\lambda(n) = 1$ when n has an even number of prime factors and $\lambda(n) = -1$ when it has an odd number of prime factors. We define $\lambda(1) = 1$. It is a completely arithmetical function obeying $\lambda(mn) = \lambda(m)\lambda(n)$ for any two positive integers m , n.

¶This was first proved by Hardy (1914).

We shall consider subsets of positive integers such as $\{n_1, n_2, n_3, n_4, ...\}$ arranged in increasing order and are such that their values of λ alternate in sign:

$$\lambda(n_1) = -\lambda(n_2) = \lambda(n_3) = -\lambda(n_4) = ... \tag{2.3}$$

It turns out that we can label such subsets with a triad of integers, which we now proceed to do. To construct such a labeling scheme, consider an example of an integer n that can be uniquely factored into primes as follows:

$$n = p_1^{e_1} p_2^{e_2} p_3^{e_3} ... p_L^{e_L} p_i p_j \tag{2.4}$$

where $p_1 < p_2 < p_3 ... < p_L < p_i < p_j$ are prime numbers and the $e_k, k \in \{1, 2, 3, ..., L\}$ are the integer exponents of the respective primes, and p_L is the largest prime with exponent exceeding 1, the primes appearing after p_L will have an exponent of only one and there may a finite number of them, though only two are shown above. Integers of this sort, with at least one multiple prime factor are referred to here as Class I integers. In contrast, we shall refer to integers with no multiple prime factors as Class II integers. A typical integer, q, of Class II may be written

$$q = p_1 p_2 p_3 ... p_j p_L, \tag{2.5}$$

where, once again, the prime factors are written in increasing order.

We now show how we construct a labeling scheme for integer sets that exhibit the property in (2.3) of alternating signs in their corresponding λ's. First consider Class I integers. With reference to (2.4), we define integers m, p, u as follows:

$$m = p_1^{e_1} p_2^{e_2} p_3^{e_3} ... p_{L-1}^{e_{L-1}}; \quad p = p_L; \quad u = p_i p_j. \tag{2.6}$$

In (2.6), m is the product of all primes less than p_L,the largest multiple prime in the factorization, and u is the product of all prime numbers larger than p_L in the factorization. Thus the Class I integer n can be written

$$n = mp^{e_L} u \tag{2.7}$$

Hence we will label this integer n as (m, p^{e_L}, u),using the triad of numbers(m, p, u) and the exponent e_L. It is to be noted that u will consist of prime factors all larger than p, and u cannot be divided by the square of a prime number.

Consider the infinite set of integers, $P_{m;p;u}$, defined by

$$P_{m;p;u} = \{mp^2 u, mp^3 u, mp^4 u, ...\} \tag{2.8}$$

The Class I integer n necessarily belongs to the above set because $e_L \geq 2$. Since the consecutive integer members of this set have been obtained by multiplying by p, thereby increasing the number of primes by one, this set satisfies property (2.3) of alternating signs of the corresponding λ's. Note that the lowest integer of this set $P_{m;p;u}$ of Class I integers is $mp^2 u$.

We may similarly form a series for Class II integers. The integer q in (2.5) may be written $q = mpu$, with $m = p_1 p_2 p_3 ... p_j$, $p = p_L$, and $u = 1$. This Class II integer is put into the set $P_{m;p;u}$ defined by

$$P_{m;p;1} = \{mp, mp^2, mp^3, mp^4, ...\}. \tag{2.9}$$

The set containing Class II integers is distinguished by the facts that $u = 1$ for all of them, their largest prime factor is always p and none of them can be divided by the square π^2 of a prime number π such that $\pi < p$; in other words the factor m cannot be divided by the square of a prime. In this set, too, the λ's alternate in sign as we move through it and so property (2.3) is satisfied. Again, note that the lowest integer of this set $P_{m;p;1}$ is the Class II integer mp, all the others being Class I.

In what follows, we shall find it handy to refer to the set of ascending integers comprising $P_{m;p;u}$ as a 'tower'. It is important to distinguish between a tower (or set) described by a triad like (m, p, u) and an integer belonging to that set. It is worth repeating that the set or tower of Class I integers described by the label (m, p, u) is the infinite sequence $\{mp^2 u, mp^3 u, mp^4 u, ...\}$, the first element of which is $mp^2 u$ and all other members of which are $mp^k u$, where $k > 2$. A set or tower containing a Class II integer described by $(m, p, u = 1)$ is the infinite sequence $\{mp, mp^2, mp^3, ...\}$, Eq.(2.9), of which only the first element mp is a Class II integer and all other members, mp^k, where $k \geq 2$, are Class I, because the latter have exponents greater than 1. For convenient reference, we shall refer to the first member of a tower as the base integer or the base of the tower. It is also worth noting that when we refer to a triad like (m, p^k, u), where $k > 1$, we are invariably referring to the integer $mp^k u$ and not to any set or tower. Labels for sets do not contain exponents; only those for integers do. Of course, the particular integer (m, p^k, u) belongs to the set or tower (m, p, u).

Two simple examples illustrate the construction of the sets denoted by $P_{m;p;u}$:

Ex. 1: The integer 2160, which factorizes as $2^4 \times 3^3 \times 5$, is clearly a Class I integer since it is divisible by the square of a prime number—in fact there are two such numbers, 2 and 3—but we identify p with 3 as it is the larger prime. It is a member of the set $P_{16;3;5} = \{16 \times 3^2 \times 5, 16 \times 3^3 \times 5, 16 \times 3^4 \times 5, 16 \times 3^5 \times 5, ...\}$.

Ex. 2: The integer 663, which factorizes as $3 \times 13 \times 17$, is a Class II integer because it is not divisible by the square of a prime number. It belongs to the set $P_{39;17;1} = \{39 \times 17, 39 \times 17^2, 39 \times 17^3, ...\}$.

Note that two different integers cannot share the same triad.[‖] And two different triads cannot represent the same integer.[**] Thus the mapping from a triad to an integer is one-one and onto. A formal proof is in the Appendix.

The following properties of the sets $P_{m;p;u}$ may be noted:

(a) The factorization of an integer n immediately determines whether it is a Class I or a Class II type of integer.

(b) The factorization of integer n also identifies the set $P_{m;p;u}$ to which n is assigned.

(c) The procedure defines all the other integers that belong to the same set as a given integer.

(d) Every integer belongs to some set $P_{m;p;u}$ (allowing for the possibility that $u = 1$) and only to one set. This ensures that, collectively, the infinite number of sets of the form $P_{m;p;u}$ exactly reproduce the set of positive integers $\{1, 2, 3, 4, ...\}$, without omissions or duplications.

Our procedure, taking its cue from the deep connection between the zeta function and prime numbers, has constructed a labeling scheme that relies on the unique factorisation of integers into primes. In what follows, we shall recast the summation in (1.2) into one over the sets $P_{m;p;u}$.The advantage of breaking up the infinite sum over all positive integers into sums over the $P_{m;p;u}$ sets will soon become clear.

3 An alternative expression of the Liouville Function's Dirichlet series

The usefulness of this Section and the next (i.e. Sections 3 and 4) is to show that the cumulative summatory function $L(N) = \Sigma_{n=1}^{N} \lambda(n)$, can be built up by 'harmonic rectangular waves', thus providing a pictorial representation of the function $L(N)$. This pictorial view which helped us to understand the phenomena in RH, actually followed the discovery of an alternative expression for Eq(1.2) namely the representation Eq(3.10). This expression is written in terms of 'towers' which as we shall see help in our study of the properties of $L(N)$. The kinks in the rectangular waves which occur at integer values k in the argument of $\lambda(k)$ each contribute either +1 or -1 to $L(N)$ and are distributed like coin tosses and their summation is akin to the cumulative sum of N coin tosses. Eq. (3.10) has also helped in evolving the concepts of Towers described in the section preceding, and apart from this Eq(3.10) plays no crucial role in the proof of RH.

THEREFORE ON A FIRST READING THE FOLLOWING CAN BE OMITTED: SEE FOOT NOTE.[††]

We shall now implement the above partitioning of the set of all positive integers to examine the analytic properties of $F(s)$ in (1.2). We shall rewrite the sum in (1.2) into an infinite number of sums of sub-series.

We begin, however, by assuming that $Re(s) > 1$, which makes the series in (1.2) absolutely convergent, in fact it represents $\zeta(2s)/\zeta(s)$ and is well defined for $Re(s) > 1$. We will not be needing the expression for regions $Re(s) < 1$. [‡‡] We write the right hand side in sufficient detail so that the implementation of the partitioning

[‖] The integer represented by the triad (m, p^r, u), is the product $mp^r u$, which obviously cannot take on two distinct values.

[**] Suppose two different triads (m, p^r, u) and (μ, π^ρ, ν) represent the same integer, say n. Then we must have $mp^r u = \mu\pi^\rho\nu = n$.It follows that at least two numbers of the tetrad $\{m, p, r, u\}$ must differ from their counterparts in the tetrad $\{\mu, \pi, \rho, \nu\}$. Since the factorization of n is unique, this is impossible.

[††] On a first reading Sections 3 and 4 can be omitted. And one can go directly to Section 5 and after reading the proof of Theorem 1, in Section 5.1, skip the rest of this subsection and go directly to subsection 5.2, in page 12 and read till the end of the paper. Though, the last four paragraphs of Section 4 should be read to understand the Figure. Sections 3 and 4 have been included to maintain mathematical rigor: To demonstrate that expressions (1.2) and (3.10) have the same analytical continuation to the left of $Re(s) = 1$ by Littlewood's theorem.

[‡‡] Though each sub-series is convergent for $Re(s) < 1$- see Titchmarsh or Whittaker and Watson.

scheme becomes self-evident:

$$\begin{aligned} F(s) &= 1+\sum_{r=1}^{\infty}\frac{\lambda(2^r)}{2^{rs}}+\sum_{r=1}^{\infty}\frac{\lambda(3^r)}{3^{rs}}+\sum_{r=1}^{\infty}\frac{\lambda(5^r)}{5^{rs}}+\sum_{r=1}^{\infty}\frac{\lambda(2\times 3^r)}{2^s3^{rs}} \\ &+\sum_{r=1}^{\infty}\frac{\lambda(7^r)}{7^{rs}}+\sum_{r=1}^{\infty}\frac{\lambda(2\times 5^r)}{2^s5^{rs}}+\sum_{r=1}^{\infty}\frac{\lambda(11^r)}{11^{rs}}+\sum_{k=2}^{\infty}\frac{\lambda(2^k\times 3)}{2^{ks}3^s} \\ &+\sum_{r=1}^{\infty}\frac{\lambda(13^r)}{13^{rs}}+\sum_{r=1}^{\infty}\frac{\lambda(2\times 7^r)}{2^s7^{rs}}+\sum_{r=1}^{\infty}\frac{\lambda(3\times 5^r)}{3^s5^{rs}}+\sum_{r=1}^{\infty}\frac{\lambda(17^r)}{17^{rs}} \\ &+\sum_{r=1}^{\infty}\frac{\lambda(19^r)}{19^{rs}}+\sum_{k=2}^{\infty}\frac{\lambda(2^k\times 5)}{2^{ks}5^s}+\sum_{r=1}^{\infty}\frac{\lambda(3\times 7^r)}{3^s7^{rs}}+\sum_{r=1}^{\infty}\frac{\lambda(2\times 11^r)}{2^s11^{rs}} \\ &+\sum_{r=1}^{\infty}\frac{\lambda(23^r)}{23^{rs}}+\sum_{r=1}^{\infty}\frac{\lambda(2\times 13^r)}{2^s13^{rs}}+\sum_{k=2}^{\infty}\frac{\lambda(2^k\times 7)}{2^{ks}7^s}+\sum_{r=1}^{\infty}\frac{\lambda(29^r)}{29^{rs}} \\ &+\sum_{r=1}^{\infty}\frac{\lambda(2\times 3\times 5^r)}{2^s3^s5^{rs}}+\cdots \quad , \end{aligned} \tag{3.10}$$

We have explicitly written out a sufficient number of terms of the right hand side of (1.2) so that those corresponding to each of the first 30 integers are clearly visible as a term is included in one (and only one) of the sub-series sums in (3.10). On the right hand side, the second term contains the integers $2, 4, 8, 16....$; the third contains $3, 9, 27, ...$; the fourth contains $5, 25, 125, ...$; the fifth contains $6, 18, 54, ...$; sixth contains $7, 49, ...$; the seventh contains $10, 50, ...$; the eighth contains $11, 121, ...$; the ninth contains $12, 24, 48, ...$; and so on. Note that in the ninth, fifteenth, and twentieth terms the running index is deliberately switched from r to k to alert the reader to the fact that the summation starts from 2 and not from 1 as in all the other sums. (Note that, in the ninth term, the Class I integer $n = 12 = 2^2 \times 3$ is assigned to the set $P_{1;2;3} = \{2^2 \times 3, 2^3 \times 3, 2^4 \times 3, ...\}$ and not to the set $P_{4;3;1} = \{2^2 \times 3, 2^2 \times 3^2, 2^2 \times 3^3, .\}$, because the first term identifies p as 2 and u as 3 where as the second term onwards 3 has exponents, which violates our rules of precedence and would be an illegitimate assignment given our partitioning rules.)

The sub-series in (3.10) have one of two general forms:

$$\sum_{r=1}^{\infty}\frac{\lambda(m.p^r)}{m^s.p^{rs}} = \frac{\lambda(m.p)}{m^s.p^s}\,[1-\frac{1}{p^s}+\frac{1}{p^{2s}}-\frac{1}{p^{3s}}+\cdots+\frac{(-1)^X}{p^{Xs}}+\cdots]$$

or

$$\sum_{k=2}^{\infty}\frac{\lambda(m.p^k.u)}{m^s.p^k.u^s} = \frac{\lambda(m.p^2.u)}{m^s.p^{2s}.u^s}\,[1-\frac{1}{p^s}+\frac{1}{p^{2s}}-\frac{1}{p^{3s}}+\cdots+\frac{(-1)^X}{p^{Xs}}+\cdots] \tag{3.11}$$

The above geometric series occurring within square brackets in the above two equations can actually be summed (because they are convergent),(see Whittaker and Watson) but we will refrain from doing so, and (1.2) can be rewritten as

$$F(s) = \sum_m\sum_p\sum_u F^T_{m;p;u}(s) + \sum_m\sum_p F^T_{m;p;1}(s), \tag{3.12}$$

where the first group of summations pertains to Class I integers n characterized by the triad $(m, p^k, u), (k \geq 2)$ and the second group pertain to those integers which are characterized by set $(m, p^k, 1), (k \geq 1)$ the first member in the set is a Class II integer and others Class I.

In the above we have defined the function $F^T_{m;p;u}(s)$ of the complex variable s which is a sub-series involving terms over only the tower (m, p, u) for a Class I integer as follows

$$F^T_{m;p;u}(s) = \sum_{k=2}^{\infty}\frac{\lambda(mp^ku)}{m^sp^{ks}u^s}, \tag{3.13}$$

and the function $F^T_{m;p;1}(s)$ of the complex variable s which is a sub-series involving terms over only the tower $(m, p, 1)$ whose 1st term is a Class II integer as

$$F^T_{m;p;1}(s) = \sum_{r=1}^{\infty}\frac{\lambda(mp^r)}{m^sp^{rs}} \tag{3.14}$$

With the understanding that when $u = 1$ we use the function in (3.14) instead of (3.13), we may write $F(s)$ as

$$F(s) = \sum_m \sum_p \sum_u F^T_{m;p;u}(s). \tag{3.15}$$

Comparing the above Eq.(3.15) with Eq(3.10) one can easily see that each term which appears as a summation in (3.10) is actually a sub-series over some tower which we denote as $F^T_{m;p;u}(s)$ in (3.15). So we see that $F(s)$ has been broken up into a number of sub-series. The important point to note is that the λ value of each term in the sub-series changes its sign from +1 to -1 and then back to +1 and -1 alternatively. Therefore if the starting value of λ at the base was +1 then the cumulative contribution of this tower (sub series) to $L(N)$ as N, the upper bound, increases from N to $N+1$, $N+2$, $N+3$, ... will fluctuate between be 0 and 1. For some other tower whose base value of λ is -1 its cumulative contribution to L(N) will fluctuate between 0 and -1; these cumulative contributions can be represented in the form of a rectangular wave as shown in Figure 1.

We have arrived at a critical point in our paper. We have cast the original function $F(s) \equiv \zeta(2s)/\zeta(s)$ as a sum of functions of s. Since the triad (m, p^k, u) uniquely characterises all integers, the summations over m, p, k and u above are equivalent to a summation over all positive integers n, as in (1.2), though not in the order $n = 1, 2, 3, 4, ...$ The manner in which the triads were defined ensures that there are neither any missing integers nor integers that are duplicated.(See Theorems A and B in Appendix II.)

Although we did not explicitly do it, we mentioned in passing that the sum over k in (3.13)and (3.14) is readily performed since it is a geometric series (see (3.11)) that rapidly converges. This is true not merely for $Re(s) > 1$ but also as $Re(s) \to 0$. Whether $F(s)$ converges when the summation is carried out over all the towers (m, p, u) and, if so, over what domain of s is the central question that we seek to answer in the next section. The answer to which as we shall see determines the analyticity of F(s) and thus resolves the Riemannian Hypothesis. We can recast (3.15), still in the domain $Re(s) > 1$, in the form

$$F(s) = \sum_{n=1}^{\infty} \frac{h(n)}{n^s}, \tag{3.16}$$

where $h(n)$ is a function appropriately defined below.

By construction, every n in the above summation can be written as

$$n = \mu \pi^\rho \nu, \tag{3.17}$$

where μ, π, and ρ are positive integers, π is the largest prime in the factorization of n, with either (i) an exponent $\rho \geq 2$, and ν is the product of primes larger than π but with exponents equal to 1 (for Class I integers) or (ii) it is the largest prime factor with $\rho = 1$ and $\nu = 1$ (for Class II integers).

We define $h(n)$ as follows:

$$\begin{aligned}
h(n) &= \lambda(mp^k u) \quad \text{if } \mu = m \text{ and } \pi = p \text{ and } \nu = u \neq 1 \text{ and } \rho = k > 1 && (3.18a)\\
h(n) &= \lambda(mp^k) \quad \text{if } \nu = u = 1 \text{ and } \pi = p \text{ and } \rho = k \geq 1 && (3.18b)\\
h(1) &= 1 \quad \text{by definition.} && (3.18c)
\end{aligned}$$

Note for all $n > 1$, (3.18a) and (3.18b) taken together, defines the $h(n)$ for all Class I and Class II integers n. The factors $m^s p^{ks} u^s$ and $m^s p^{ks}$ in the denominators of (3.13) and (3.14) are simply n^s, where n is the integer characterized by the (m, p^k, u) triad (with $u = 1$ in the latter case).

4 Representation of the summatory Liouville function L(N)

We are now in a position to examine the summatory Liouville function $L(N)$ and to depict the sum for any given finite N, as arising from individual contributions from 'rectangular waves'.

To do all this systematically, we will explicitly illustrate the process starting from $N = 1, 2, 3...$ up to $N = 15$. Each of these numbers is factored and expressed uniquely as a triad. The N=1 is a constant term, which is the trivial $(1, 1, 1)$, then the next number $N = 2 = (1, 2, 1)$, is contained in the tower shown below the one corresponding to $N = 1$; and $N = 3 = (1, 3, 1)$, is the tower below the previous; $4 = (1, 2^2, 1)$ however 4 is already contained in the tower $(1, 2, 1)$ as its second member; the next N's: $5, 6, 7$, give rise to the new towers $(1, 5, 1), (2, 3, 1), (1, 7, 1)$; 8 of course is the third member of the old tower $(1, 2, 1)$ similarly 9 is the 2nd member of $(1, 3, 1)$. After this the new towers which make their appearance are: $10 = (2, 5, 1), 11 = (1, 11, 1), 13 = (1, 13, 1), 14 = (2, 7, 1)$ and $15 = (3, 5, 1)$. Figure 1 shows these and numbers up to N=30.

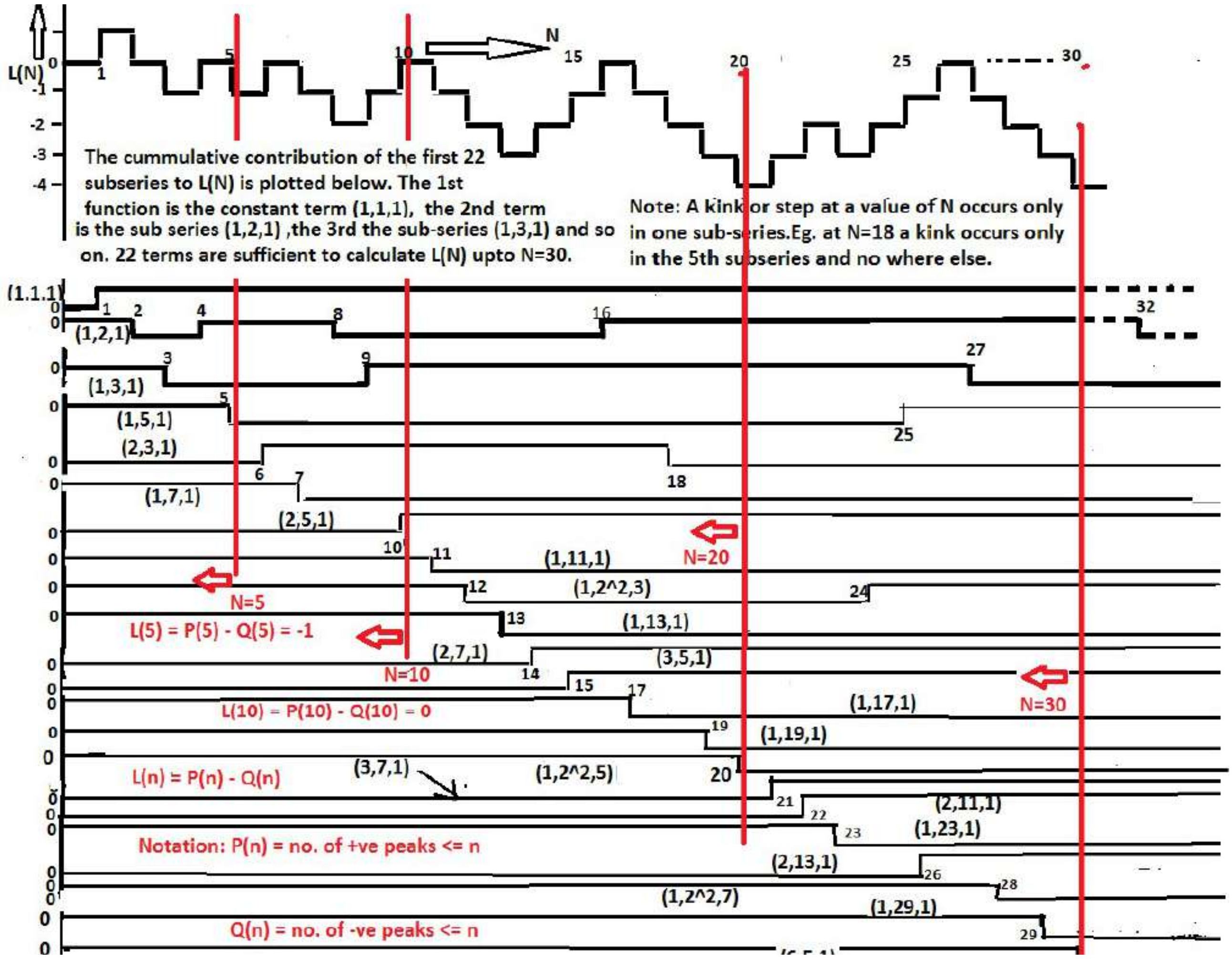

Fig. 1. The cumulative sum, L(N) (see top), is obtained by 'filling up' slots in various towers from the bottom up until we have exhausted all N integers.

Now each tower (m, p, u) contributes to $L(N)$ (consider N fixed in the following) according to the following rules:

(i) A particular tower will contribute only if its base number is less than or equal to N, i.e. $m.p.u \leq N$

(ii) And the contribution C to $L(N)$ from this particular tower will be exactly as follows:

Case A; Class II integer $(u = 1)$

$C = \Sigma_{r=1}^{R} \lambda(m.p^r.1)$, where R is the largest integer such that $m.p^R \leq N$

Case B; Class I integer $(u > 1)$

$C = \Sigma_{k=2}^{K} \lambda(m.p^k.u)$, Where K is the largest integer such that $m.p^K.u \leq N$

Now since each successive λ changes sign from $+1$ to -1 or vice a versa, the contributions of each tower can be thought of as a rectangular wave of ever-increasing width but constant amplitude -1 or $+1$, see Figure 1.

To find the value of L(N), (N fixed), all we need to do is count the jumps of each wave: as we move from N=0 a jump upwards is called a positive peak, a jump downwards is a negative peak. Draw a vertical line at N, we are assured that it will hit one and only one peak (positive or negative) in one of the sub-series; then count the total number of positive peaks P(N) and negative peaks Q(N), of the waves on and to the left of this vertical line, then $L(N) = P(N) - Q(N)$; the reason for this rule will be clear after the next section.

For an example, take $N = 5$. There is a positive peak for the constant term (1,1,1), the next wave (1,2,1) contributes one negative peak (at 2) and a positive peak (at 4), the wave (1,3,1) contributes a -1 peak (at 3) and (5,1) contributes a -1 peak (at 5). Thus a total of three negative peaks and two positive peaks add up to give $L(5) = -1$, which is of course correct. Now if we take $N = 10$, and draw a vertical line at N=10, looking at this line and to its left we see that there are additionally three positive and two negative peaks thus adding this contribution of $+1$ to the previously calculated value $L(5)$ we get $L(10) = 0$. (Two red vertical lines just just beyond N=5 and N=10 are drawn for convenience.) Now if we wish to compute L(15) we see that there are three more negative peaks and two positive peaks thus giving a value $L(15) = -1$. Counting the peaks further on it is easy to check that $L(N)$ is correctly predicted for every value of N up to 30 and in particular, $L(20) = -4$, $L(26) = 0$ and $L(30) = -4$.

In summary, to calculate L(N) we merely need to count the negative and positive peaks of the waves on N and to the left of N. In the figure we have drawn a number of waves and labeled the tower to which each belongs using a triad of numbers. They are sufficient for one to easily calculate L(N) up to N=30 and check them out by comparing the numbers with the plot of L(N) shown on the top of the figure.

We turn to a more fundamental point: We show, in Section 6, that, for sufficiently large N (see Appendix IV), the distribution of the value of L(N) is equivalent to that obtained from summing the distribution of N coin tosses.

5 Determination of Analyticity of $F(s)$ using Littlewood's Theorem

We now utilize a technique introduced by Littlewood (1912), to examine the analyticity of the function $F(s)$. In this, we follow the treatment of Edwards (1974, pp. 260-261).

We have seen that there are two equivalent expressions $F(s)$ viz. Eqs.(1.2) and (3.10) both of which are given in the form of a series and are absolutely convergent in the region $Re(s) > 1$. We will therefore follow the following two procedures:

(i) By using Littlewoods technique we will analytically continue Eq (1.2), which is convergent for $Re(s) > 1$ to regions $Re(s) < 1$ and then see that his theorem determines a condition on L(N) for N large, for RH to be true.

(ii) Similarly instead of using Eq (1.2) we use the equivalent (3.10) and use Littlewood's technique to analytically continue Eq (3.10) which is convergent for Re(s) ¿ 1 to regions Re(s) ¡ 1 . This also gives a same condition as (a) on L(N) for N large for RH to be true. But this time the condition can be interpreted by a FIGURE. And the FIGURE reveals a clear analogy with coin tosses. Strictly speaking our treatment (ii) is redundant except for the understanding of the connection of L(N) with coin tosses. In fact for the rest of the paper we do not need Eq. (3.10) or the Figure, except that the concept of Towers and the factorizations of integers and the determination of their membership to different towers would be needed to prove several theorems.

5.1 Littlewood's theorem applied to F(s) viz. Eqs.(1.2) & (3.10)

We have seen that (1.2) we define:

$$F(s) = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}, \tag{5.19}$$

Similarly the alternative expression (3.10) written in the form Eq. (3.16) is:

$$F(s) = \sum_{n=1}^{\infty} \frac{h(n)}{n^s}, \tag{5.20}$$

with the definition given in (3.18).

Since both of the above expressions are similar in form we use the following generic expression for the purpose of analysis:

$$F(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}, \tag{5.21}$$

where $g(n)$ can mean $\lambda(n)$ or $h(n)$ as the case may be.

The above series (5.19) can be expressed as the integral

$$F(s) = \int_0^{\infty} x^{-s} dG(x), (Re(s) > 1), \tag{5.22}$$

where $G(x) = \int_0^x dG$ is a step function that is zero at $x = 0$ and is constant except at the positive integers, with a jump of $g(n)$ at n. The value of $G(n)$ at the discontinuity, at an integer n, is defined as $(1/2)[G(n-\epsilon) + G(n+\epsilon)]$, which is equal to $\sum_{j=1}^{n-1} g(j) + (1/2)g(n)$. Assuming $Re(s) > 1$, integration by parts yields

$$F(s) = \int_0^\infty d[x^{-s}G(x)] - \int_0^\infty G(x)d[x^{-s}] \qquad (5.23)$$

$$= \lim_{X\to\infty} \quad [X^{-s}G(X) + s\int_0^X G(x)x^{-s-1}dx$$

$$= s\int_0^\infty G(x)x^{-s-1}dx, \qquad (5.24)$$

where the last step follows from the fact that $\mid G(X) \mid \leq X$, which implies that $X^{-s}G(X) \to 0$ as $X \to \infty$. We further observe, following Littlewood (1912), that as long as $G(X)$ grows less rapidly than X^a for some $a > 0$, the integrals in (5.21) and in the line preceding it converge for all s in the half-plane $Re(a - s) < 0$, that is, for $Re(s) > a$. By analytic continuation, $F(s)$ converges in this half-plane. Since this result will be important in what follows, we record it here.

Theorem [Littlewood (1912)]: When $G(X)$ grows less rapidly with X than X^a for some $a > 0$, $F(s)$ is analytic in the half-plane $Re(s) > a$.

We have obtained the above generic result for the analytic continuation of the function given by Eq(5.21). We will now apply it for the case (i) i.e. Eq. (5.19),in this case $g(n) \equiv \lambda(n)$ and $X \equiv N$ thus making $G(X) \equiv L(N)$. Thus the above Littlewood's Theorem becomes the condition for the analytic continuation of(5.19) and which is now restated to read:

Theorem 1 [Littlewood (1912)]: When $L(N)$ grows less rapidly with N than N^a for some $a > 0$, $F(s)$ is analytic in the half-plane $Re(s) > a$.

We shall now demonstrate that the sufficient condition stated in Theorem 1 is satisfied for a specific value of a that settles the Riemann Hypothesis. (It will turn out that $a = 1/2$).

Now before devoting the rest of the paper to show that the above condition holds for RH. We will use the analysis for the analytical continuation of (5.20) i.e. Case (ii).

Hence our definition of $G(N)$ becomes

$$G(N) = \sum_{n=1}^{N} g(n), \qquad (5.25)$$

and we may rewrite $G(N)$ as

$$G(N) = \sum_m \sum_p \sum_u \sum_k \left[(1 - \delta_{u,1}).(1 - \delta_{k,1})\, \lambda(mp^k u) + \delta_{u,1}\lambda(mp^k)\right], \qquad (5.26)$$

where $\delta_{u,1}$ and $\delta_{k,1}$ are Kronecker deltas (e.g. $\delta_{u,1} = 1$ if $u = 1$ and 0 otherwise). The summations over m, p, k, and u in (5.26) are undertaken with the understanding that the triads (m, p^k, u) will only include integers $n \leq N$. Since the summation over k is over an individual tower(if we keep (m,p,u) fixed we can write(5.26) as

$$G(N) = \sum_m \sum_p \sum_u F^T_{m,p,u}(s = 0), \qquad (5.26b)$$

This is nothing but Eq.(3.15) evaluated from each subseries $F^T_{m,p,u}(s)$ by making $s \to 0$.

Of course, what we have called $G(N)$ is really the summatory Liouville function, $L(N)$, defined earlier by (1.2b), because each integer n occurs only once in the r.h.s. of (5.26b) as an argument of λ i.e. $\lambda(n)$, Therefore the $G(N)$ is really $L(N)$, hence

$$L(N) = \sum_{n=1}^{N} \lambda(n). \qquad (5.24)$$

From now on, we revert to the original definitions of the sequence $h(n) \equiv \lambda(n)$ and $G(N) \equiv L(N)$ as defined in Eq. (1.2) but we may write them in the forms derived in Section 2 using triads.

5.2 Derivation of Theorems concerning Factorization Sequence of $\lambda's$ and the Final Proof of the Riemann Hypothesis

In this subsection we will derive several crucial theorems concerning the sequence of $\lambda's$.

Expression (5.26) is crucial because, in the light of Theorem 1, its behaviour will determine the validity of the Riemann Hypothesis. Every term in the summation in (5.26) is either $+1$ or -1. We need to determine, for given N, how many terms contribute $+1$ and how many -1, and then determine how the sum $G(N)$ varies with N.

As we go through the list $n = 1, 2, 3, \cdots, N$, we are assigning the integers to various sets of the kind $P_{m;p;u}$. To use our terminology of towers, we shall be 'filling up' slots in various towers from the bottom up until we have exhausted all N integers. (When N increases, in general, we shall not only be filling up more slots in existing towers but also adding new towers that were previously not included.) So the behaviour of $G(N)$ is determined by how many of the numbers that do not exceed N contribute $+1$ and how many -1.

It is convenient to identify the λ of an integer by the triad which uniquely defines that integer. To avoid abuse of notation, we shall denote the value $\lambda(n)$ in terms of the λ-value of the base integer of the tower to which n belongs. We will define the λ of the base of a tower in uppercase, as $\Lambda(m, p, u)$. In other words if $n = (m, p^\rho, u)$ then it will belong to a tower whose base number is $n_B \equiv (m, p^\kappa, u)$, where $\kappa = 2\ if\ \ u \neq 1$ and $\kappa = 1\, if\ \ u = 1$. Now we define $\Lambda(m, p, u) = \lambda(n_B) = \lambda(mp^\kappa u) = \lambda(m)\lambda(p^\kappa)\lambda(u)$, since the λ of a product of integers is the product of the λ of the individual integers. Of course, once we know $\lambda(n_B)$ we will know the λ of all other numbers belonging to the tower because they alternate in sign.

To determine the behaviour of $G(N)$, the following theorem is important.

Theorem 2: For every integer that is the base integer of a tower labeled by the triad (m, p, u), and therefore belonging to the set $P_{m;p;u}$, there is another unique tower labeled by the triad (m', p, u) and therefore belonging to the set $P_{m',p,u}$ with a base integer for which $\Lambda(m', p, u) = -\Lambda(m, p, u)$.

Proof:

Let us write the integers at the base of a tower in the form $n = mp^\rho u$ described by the triad (m, p, u), where we shall assume that $\rho = 2$ if $u \neq 1$ and $\rho = 1$ if $u = 1$. These correspond to the smallest members of sets of Class I and Class II integers, respectively, which are the integers of concern here. In the constructions below, we shall multiply (or divide) m by the integer 2. Since 2 is the lowest prime number, such a procedure does not affect either the value of p or u in an integer and so we can hold these fixed.

We begin by excluding, for now, triads of the form $(1, p, 1)$, integers which are single prime numbers. We allow for this in Case 3 below.

Case 1: Suppose m is odd. We choose $m' = 2m$, then
$\Lambda(m', p, u) = \Lambda(2m, p, u) = -\Lambda(m, p, u)$. We may say that (m, p, u) and (m', p, u) are 'twin' pairs in the sense that their Λs are of opposite sign. Note that (m, p, u) and (m', p, u) are integers at the base of two different towers; they are not members of the same tower. (Recall that the members of a given tower are constructed by repeated multiplication with p.)

Case 2: Suppose m is even. In this case, we need to ascertain the highest power of 2 that divides m. If m is divisible by 2 but not by 2^2, assign $m' = m/2$. (So $m = 6$ gets assigned to $m' = 3$, and $m = 3$, by Case 1 above, gets assigned to $m' = 6$.) More generally, suppose the even m is divisible by 2^k but not by 2^{k+1}, where k is an integer. Then, if k is even, assign $m' = 2m$; and if k is odd, assign $m' = m/2$. (So $m = 12 = 2^2 \times 3$ gets assigned to $m' = 2^3 \times 3 = 24$. And, in reverse, $m = 24 = 2^3 \times 3$ gets assigned to $m' = 24/2 = 12$.)

Thus for odd m the following sequence of pairs (twins) hold:
(m, p, u) and $(2m, p, u)$ are twins at bases of different towers having λs of opposite signs,(this is Case 1),
$(2^2m, p, u)$ and $(2^3m, p, u)$ are twins at bases of different towers having λs of opposite signs,
$(2^4m, p, u)$ and $(2^5m, p, u)$ are twins at bases of different towers having λs of opposite signs,
and so on.

Case 3: Now consider the case where the triad describes a prime number; that is, it takes the form $(1, p, 1)$. For the moment, suppose this prime number is not 2. In this case, where $m = u = 1$, we simply assign $m' = 2$. Clearly,
$\Lambda(2, p, 1) = -\Lambda(1, p, 1)$, and the numbers $(2, p, 1)$ and $(1, p, 1)$ are at the bases of different towers.

Case 4: Finally, consider the case where the triad describes a prime number and the prime number is 2; that is, the integer $(1, 2, 1)$, for which $\Lambda(1, 2, 1) = -1$. We match this prime to the integer 1. By definition $\lambda(1) = \Lambda(1, 1, 1) = 1$. Thus the first two integers have opposite signs for their values of λ. $\square$

So, in partitioning the entire set of positive integers, the number of towers that begin with integers for which $\lambda = -1$ is exactly equal to those that begin with integers for which $\lambda = +1$.

Thus, Theorem 2 immediately gives the following result:

Theorem 3: In the set of all positive integers, for every integer which has an even number of primes in its factorization there is another unique integer, (its twin), which has an odd number of primes in its factorization.

The consequence of the above theorem follows not just from that each integer has a unique twin whose λ-value is of the opposite sign, but also from from the context that these lie in an alternating sequence. That is, not only are the bases of two uniquely-paired towers twins, the next higher number in the first tower is the twin of the next higher number in the second tower, and so on. Thus every integer with $\lambda = \pm 1$ in a unique tower is twinned uniquely in alternating sequence with the integers with $\lambda = \mp 1$ in another unique tower, ensuring that the proportions of numbers with $\lambda = +1$ and $\lambda = -1$ are equal over the entire natural number system. Alternately, it may be argued that each Tower, which is an ordered infinite sub-set of the natural number system, is itself equally partitioned into members with $\lambda = +1$ and $\lambda = -1$ by their alternating sequence in that order. As the natural number system (excluding 1) comprises only such Towers, it too is so equally partitioned, and has equal proportions of numbers with $\lambda = +1$ and $\lambda = -1$.

Thus we have shown:*

Theorem 3B: If n is an arbitrary positive integer,

$$Prob[\lambda(n) = +1] = Prob[\lambda(n) = -1] = 1/2. \tag{19}$$

The result in Theorem 3B is a necessary condition for RH to be valid; it is not sufficient.† The condition that is equivalent to proving RH ((Littlewood (1912), Edwards (1974)) is the following:

$$\frac{L(N)}{N^{\frac{1}{2}+\epsilon}} = 0,\ as\ N \to \infty, \tag{20}$$

for any $\epsilon > 0$. We describe how this equivalence is formally proved below.

1. We compare the $L(N)$ series to $X(N)$, the distance travelled in N unit steps in a standard *random walk*, which can be represented as:

$$X(N) = \sum_{n=1}^{N} c(n) \tag{21}$$

   where the $c(n)$'s are independent random numbers with an equal probability of being either +1 or -1, i.e., "coin-tosses". It is a well-known result (see Chandrasekhar (1943)) that the expected value of $|X(N)|$, for large N, is

$$lim_{N\to\infty} E(|X(N)|) = C_0 N^{1/2} \tag{22}$$

   The further line of advance of the proof is to show now that Equation (22) applies to $L(N)$ as well, and so proves Equation (20), and thereby the RH. To do this we have to prove that the $\lambda(n)$'s in the $L(N)$ series are essentially "coin-tosses", for large n.

2. To show that the λ's behave as coin tosses, we have to show that (i) their probabilities of being either +1 of -1 are equal, as was proved by Theorem 3B, further (ii) we have to show that the λ's appearing in the natural sequence, $n = 1, 2, 3, ...$, are independent of each other — i.e., that the value of $\lambda(n)$ has no influence on the value of $\lambda(n+1)$, say. This seems counter-intuitive, as the λ's are obviously deterministically linked. Nevertheless, their independence in the natural sequence is shown by two different approaches:
   (a) In Appendix III, it is proved that the sequence of $\lambda(n), n = 1, 2, 3, ...$ is non-cyclic. This would preclude any dependence of the type
   $\lambda_n = f(\lambda_{n-1}, \lambda_{n-2}, \lambda_{n-3}, ..., \lambda_{n-M})$
   because any finite series of +1's and -1's of length M would have a finite number of permutations P, so the series $\lambda_{n-1}, \lambda_{n-2}, \lambda_{n-3}, ..., \lambda_{n-M}$ must repeat itself after atmost P numbers, and thereafter become cyclic *if* such a dependence relationship exists between the λ's. The non-cyclic nature of $L(N)$ conforms also to the notion of randomness in Knuth (1968, Ch. 3).

*These Theorems 3 and 3B, were later given new, alternate proofs, which only need induction and the starting premise that every odd integer starting from 1, has a unique successor integer, which is even and with which it forms a unique 'Pair' and that the even integer in every such Pair, has an odd integer which is its predecessor and its 'partner', the successor of an even integer is an odd integer not its 'partner'. See: K. Eswaran, (April 2018)

†Borwein et al (2006, p. 48) claim that the result in Theorem 3 is equivalent to a proof of RH.

(b) In Appendix IV, another approach is taken. It is shown that from merely these two rules: $\lambda(p) = -1$ for a prime p, and $\lambda(pq) = \lambda(p) \times \lambda(q)$, where q is any integer, the entire sequence of λ's for $n = 1, 2, 3, ...,$ can be obtained without determining the number of prime factors of n. It is then argued that for any two integers $n > m$, $\lambda(n)$ is dependent on $\lambda(m)$, if the latter is required to find the former, and independent if not. It is then shown that, as $n \to \infty$, any finite sequential strip of λ's will be independent of each other, thus essentially making them equivalent to coin-tosses.

3. With Theorem 3B and Appendices III and IV, we have proved above that the $L(N)$ series is *one* realization of a random walk. We then invoke (towards the end of Section 5 in [5]) Khinchin and Kolmogorov's law of the iterated logarithm to show that the maximum deviation d_N, from the one-half power-law expectation, in the exponent of $|X(N)|$ for any individual random walk tends monotonically to zero as $N \to \infty$. So (22) also holds for $|L(N)|$. Therefore for any chosen $\epsilon > 0$ in (20), the statement for the validity of RH will be satisfied and the Riemann Hypothesis is proved.
4. In Appendix V, starting from Littlewood's *ansatz*, that $L(N) \sim N^a$, for $N \to \infty$, we argue that the statistics of the λ's must become "self-similar", i.e., independent of N for large N. This principle yields us the value of $a = \frac{1}{2}$, which again would satisfy (20). This 'physicist's proof' of the RH, is separate from the argument in the main paper, and may be treated as an interesting addendum to it.
5. Finally, in Appendix VI, we confirm by considering the λ's from $n = 1$ to 176 *trillion*, that their sequence is statistically indistinguishable (using the χ^2 statistical test) from coin-tosses over the entire set of numbers considered, and also when it is partitioned in smaller sections. While this is merely a 'verification', not a 'proof', this fact has not been reported in literature, and by itself, requires an explanation (which we have provided) given its surprising nature.

With Theorem 3B and Appendices III and IV, we have proved that the $L(N)$ series is a random walk. We formally confirm this below.

Theorem 4: The summatory Liouville function, $L(N) = \sum_{n=1}^{N} \lambda(n)$ *has the following asympotitic behaviour:* $|L(N)| \leq C_0 N^{\frac{1}{2}+d_N}$ *as* $N \to \infty$.

Proof: Theorem 3B gives $Pr(\lambda(n) = +1) = Pr(\lambda(n) = -1) = 1/2$, where Pr denotes probability. Given the results in Appendicies III and IV, the λ-values behave like 'ideal coin' tosses, where $\lambda(n) = +1$ as head and $\lambda(n) = -1$ as tail, and $L(N)$ is the cumulative result of N successive coin tosses, and is equivalent to the distance $X(N)$ moved in a random-walk with N unit steps. Chandrasekhar (1943) has shown that, for a random walk of N steps, Expectation$(|X(N)|) = C_0 N^{\frac{1}{2}}$ *as* $N \to \infty$. The quantity d_N (≥ 0) seen above is the maximum deviation from expectation of an individual random walk of N steps. QED

We conclude this section by estimating the 'width' of the Critical Line, the region around $Re(s) = 1/2$ in which the non-trivial zeros of $\zeta(s)$ must lie. Invoking Littlewood's Theorem (Sec.5), we deduce that $F(s) \equiv \zeta(2s)/\zeta(s)$ is analytic in the region $a = 1/2 + d_\infty < s < 1$ (where $d_\infty \equiv lim_{N\to\infty} d_N$). This implies $\zeta(s)$ has no zeros in the same region. But Riemann had shown by using symmetry arguments‡ that if $\zeta(s)$ has no zeros in the latter region then it will have no zeros in the region $0 < s < 1/2 - d_\infty$; taking both these results together we are lead to the conclusion that all the zeros can only lie in the $1/2 - d_\infty < Re(s) < 1/2 + d_\infty$.

It is interesting that the law of the iterated logarithm enunciated by Kolmogorov (1929), also see Khinchine (1924), gives an expression for d_N. The statement of the law, adapted to the present context, is: *Let* $\{\lambda_n\}$ *be independent, identically distributed random variables with means zero and unit variances. Let* $S_N = \lambda_1 + \lambda_2 + \ldots + \lambda_N$. *The limit superior (upper bound) of* S_N *almost surely (a.s.) satisfies*

$$Lim\ Sup \frac{S_N}{\sqrt{2N\, log\, log\, N}} = 1 \quad \text{as} \quad N \to \infty$$

Now, from Theorem 4 we have written that if we consider the $\lambda's$ as "coin tosses" one can write $L(N) = \lambda_1 + \lambda_2 + \ldots + \lambda_N \leq C_0 N^{\frac{1}{2}+d_N}$ (as $N \to \infty$) (since we are interested in only the behaviour for large N we henceforth ignore the constants). Comparing this expression with the one above we see that one

‡He did this first by defining an associated xi function: $\xi(s) \equiv \Gamma(s/2)\pi^{s/2}\zeta(s), \Gamma(s)$ is the Euler Gamma function, then showed that this xi function has the symmetry property $\xi(s) = \xi(1-s)$ which in turn implied that that the zeros of $\zeta(s)$ (if any) which are not on the critical line will be symmetrically placed about the point s=1/2, ie.if $\zeta(\frac{1}{2} + u + i\sigma)$ is a zero then $\zeta(\frac{1}{2} - u - i\sigma), (0 < u < 1/2)$, is a zero see Whittaker and Watson page 269.

can write $N^{\frac{1}{2}+d_N} \sim \sqrt{N\,log\,log\,N}$ thus yielding an expression§ for $d_N = \frac{log\,log\,log\,N}{2\,log\,N}$. We see that $d_N \to 0$ as $N \to \infty$. So the equivalent statement Equation (20) will be satisfied for any chosen $\epsilon > 0$. Further d_∞ ($\equiv lim_{N\to\infty} d_N$) is the half-width of the critical line. Since this is zero, we conclude that all the non-trivial zeros of the zeta function must lie strictly on the critical line. Thereby, the Riemann Hypothesis is proved.

6 Conclusions

In this paper we have investigated the analyticity of the Dirichlet series of the Liouville function by constructing a novel way to sum the series. The method consists in splitting the original series into an infinite sum over sub-series, each of which is convergent. It so turns out each sub-series is a rectangular function of unit amplitude but ever increasing periodicity and each along with its harmonics is associated with a prime number and all of them contribute to the summatory Liouville function and to the Zeta function. A number of arithmetical properties of numbers played a role in the proof of our main theorem, these were: the fact that each number can be uniquely factorized and then placed in an exclusive subset, where it and its other members form an increasing sequence and factorize alternately into odd and even factors and thus have equal proportions of numbers with $\lambda = +1$ and -1; and each subset can be labelled uniquely using a triad of integers which in their turn can be used to determine all the integers which belong to the subset. This helped us to show that for every integer that has an even number of primes as factors (multiplicity included), there is an integer that has an odd number of primes. This provides a proof for the long-suspected (Denjoy 1931) but unproved conjecture—until now—that the Riemann Hypothesis has a connection with the coin-tossing problem. Further, it has now been revealed that the randomness of the $\lambda(n)$'s in the natural sequence¶ is the reason that the non-trivial zeros of the Zeta function all lie on the critical line: $Re(s) = 1/2$.

7 DEDICATION

I dedicate this paper to my teachers: Mr John William Wright of Bishop's School Poona, Prof. S.C. Mookerjee of St. Aloysius' College Jabalpur, Prof. P.M.Mathews of Univerity of Madras, Mr. D.S.M. Vishnu of BHEL R& D Hyderabad and to my first teachers - my parents. All of them lived selfless lives and nearly all are now long gone: May they live in evermore.

Acknowledgments: I thank my wife Suhasini for her unwavering faith and encouragement. I thank my brothers Mukesh and Vinayak Eswaran, Dr. S.V Ramanan and Prof. George Reuben Thomas for carefully going over the manuscript and for their suggestions.

I acknowledge with grateful thanks my email correspondence with Prof. Sir Michael Berry, Professor of Physics (emeritus) Univ. of Bristol and Prof. German Sierra, Inst. of Theoretical Physics, UAM-CSIC, Madrid, which has helped explain the various intricate arguments expounded in the Extended Abstract of this paper.

I also thank the support of the management of SNIST, viz. Dr. P. Narsimha Reddy and Dr. K.T. Mahi, for their constant support, and my departmental colleagues and the HOD Dr. Aruna Varanasi for providing a very congenial environment for my research.I sincerely thank Mr.Abel Nazareth of Wolfram Research for providing me a version of Mathematica which made some of the reported calculations in Appendix VI, possible.

8 APPENDIX I: Scheme of partitioning numbers into sets

Our scheme of partitioning numbers into sets is as follows:

(a) Scheme for Class I integers:

Let us say $n = p_1^e p_2^f p_3^g ... p_m^h p_L^k p_j p_t$, then it will have at least one prime which has an exponent of 2 or above and among these there will a largest prime p_L whose exponent is atleast 2 or above. Such a prime will always exist for a Class I number. Then by definition the number to the right of p_L is either 1 or is a product of primes with exponents only 1. Now multiply all the numbers to the left of $p_L$and call it m i.e. $m = p_1^e p_2^f p_3^g ... p_m^h$ and the

§ $N^{\frac{1}{2}+d_N} \sim \sqrt{N\,log\,log\,N} = e^{log\,\sqrt{N\,log\,log\,N}} = e^{\frac{1}{2}log\{N\,log\,log\,N\}} = e^{\frac{1}{2}log\,N + \frac{1}{2}log\,log\,log\,N} = N^{\frac{1}{2}}\,e^{\frac{1}{2}log\,log\,log\,N}$ which then implies $e^{\frac{1}{2}log\,log\,log\,N} = N^{d_N} = e^{d_N\,log\,N}$ thus giving $d_N = \frac{log\,log\,log\,N}{2\,log\,N}$

¶God seems to have "played dice" at least once, when he created the natural number system!

product of numbers to the right of $p_L$as u i.e. $u = p_j p_t$.Now this triad of numbers m, p_L, u will be used to label a set,note $n = m.p_L^k.u$ Let us define the set $P_{m;p_L;u}$:

$$P_{m;p_L;u} = \{m.p_L^2.u,\ m.p_L^3.u,\ m.p_L^4.u,\ m.p_L^5.u,\ m.p_L^6.u,\ m.p_L^7.u,\ ....\} \qquad (A1)$$

Obviously $n = m.p_L^k.u$ which has $k \geq 2$ belongs to the above set. Also notice the factor involved in each number increases by a single factor of $p_L$therefore the λvalues of each member alternate in sign:

$$\lambda(m.p_L^2.u) = -\lambda(m.p_L^3.u) = \lambda(m.p_L^4.u) = -\lambda(m.p_L^5.u) = \lambda(m.p_L^6.u) = ...(A2)$$

In this paper ALL sets defined as $P_{m;p;u}$will have the property of alternating signs of λ Eq. (A1). Note in the above set containing only Class I integers m will have only prime factors which are each less than p_L.

Let us consider various integers:

Ex 1. Let us consider the integer 73573500; this is factorized as $2^2.3.5^3.7^3.11.13$ and since this is a Class I integer, and $p_L = 7$ because 7 is the highest prime factor whose exponent is greater than one. $p_L = 7$ $m = 2^2.3.5^3$ and $u = 11.13 = 143$ and therefore 73573500 is a member of the set $P_{1500;7;143}$

$$P_{1500;7;143} = \{1500.7^2.143,\ 1500.7^3.143,\ 1500.7^4.143,\ 1500.7^5.143,\ ...\}$$

Ex 2. Now let us consider the simple integer: 3^4 this is a class I integer and belongs to $P_{1;3;1} = \{3, 3^2,\ 3^3,\ 3^4,\ 3^5,\ 3^6,\ ......\}$

Ex 3. Let us consider the integer 663 this is factorized as: 3.13.17 and is a Class II integer as there no exponents greater than 1, and 663 = 3.13.17 and since 17 is the highest prime number we put this in the set:

$$P_{39;17;1} = \{39.17,\ 39.17^2,\ 39.17^3,\ 39.17^4, ...\}.$$

NOTE: If a tower has a Class II integer then it will appear as the first (base) member, all other numbers will be Class I numbers.

Ex 4. Let the integer be the simple prime number 19, we write:

$$19 \,\epsilon\, P_{1;19;1} = \{19,\ 19^2,\ 19^3,\ 19^4,\ .....\}$$

Ex 5. Let the integer be 4845 this is factorized as3.5.17.19 since this is a Class II integer we see $m = 3.5.17 = 255$, $p = 19$, $u = 1$ and the set which it belongs is

$$P_{255;19;1} = \{255.19,\ 255.19^2,\ 255.19^3,\ 255.19^4,\ 255.19^5, ...\}$$

9 APPENDIX II: Theorems on representation of integers and their partitioning into sets.

Theorem A: Two different integers cannot have the same triad (m, p^k, u)

Let a and b be two integers which when factored according to our convention are $a = n.q^g.v$ and $b = n'.q'^h.v'$, and let us consider only Class I integers u, v and v' are all > 1.

If they are both equal to the same triad (say) $(m, p^k.u)$. Then $m.p^k.u = n.q^g.v = n'.q'^h.v'$. Consider the first two equalities $m.p^k.u = n.q^g.v$, which means p is the largest prime with $k > 1$ on the l.h.s. Similarly q is the largest prime with exponent $g > 1$ on the r.h.s. Now if $p > q$ this means p^k must divide v, but this cannot happen since v cannot contain a prime greater than q with an exponent $k > 1$. Now if $p < q$ then q^g must divide u but this again cannot happen since u cannot contain an exponent $g > 1$. So we see $p = q$, and $k = g$. But once again unique factorization would imply, since u contains all prime factors larger than p and v must contain only prime factors larger than $q(= p)$, the only possibility is $u = v$, but this also makes $m = n$. That is, the triad of a is (m, p^k, u). Similarly equating the second and third equalities $n.q^g.v = n'.q'^h.v'$ and using similar arguments we see $n = n'$, $q = q'$, and $v = v'$; that is, $a = b$. The same logic can be used to prove the theorem for class II integers when $u = v = v' = 1$. QED.

Theorem B: Two different triads cannot represent the same integer.

If there are two triads (m, p^e, u) and (m', r^s, u') and represent the same integer say a which can be factorized as $a = n.q^g.v$. Where the factorization is done as per our rules then we must have $m.p^e.u = n.q^g.v$ by using exactly similar arguments as above(in Theorem A) we conclude that we must have $m = n, p = q, e = g$ and $u = v$; similarly imposing the condition on the second triad $m'.r^s.u' = n.q^g.v$, we conclude $m' = n, r = q, s = g$ and $u' = v$; thus obtaining $m = m', p = r, e = s$ and $u = u'$ this means the two triads are actually identical. QED

10 APPENDIX III: Non-cyclic nature of the factorization sequence

It is a necessary condition in the tosses of an ideal coin that the results are not cyclic asymptotically, namely the results cannot form repeating cycles as the number of tosses becomes large.

Definition

Let n_k be the number of primes, repetitions counted, in the factorization of a positive integer k. We call $\{n_1, n_2, ..., n_k, ...\}$ the factorization sequence.

Note: $\lambda(k) = +1$ if n_k is even and $\lambda(k) = -1$ if n_k is odd.

Theorem The factorization sequence is asymptotically non-cyclic.

Proof: The result follows from this claim:

Claim. The sequence $\lambda(1), \lambda(2), \lambda(3), ..., \lambda(n), ...,$ is asymptotically non-cyclic.

If the claim is not true there would exist an integer $t, t \geq 0$, so that the sequence is cyclic (after $\lambda(t)$), with cycle length σ.

By Theorem 3, the number of positive integers with even number of prime factors (counting multiplicities) equals the number of positive integers with odd number of prime factors (counting multiplicities). Therefore, the λ's in each cycle must sum to zero as do the first t λ's before the cycles start.

Then $L(N) \leq max\{t/2, \sigma/2\}$.

Now we use Littlewood's Theorem 1 and noting that in (5.21) $G(x) \equiv L(x)$, we substitute the maximum value of $L(x)$ as $x \to \infty$, viz. $\mid L(x) \mid = \sigma/2$, and thus deduce that (5.21) will always converge provided $0 < s$. Since, $\mid L(x) \mid \leq \sigma/2$, $L(x)$ indeed grows less rapidly than x^a for all $a > 0$, satisfying the condition in Theorem 1. This means that we should be able to analytically continue $F(s) \sim \zeta(2s)/\zeta(s)$ leftwards from $Re(s) = 1$ to $Re(s) = 0$, contradicting Hardy (1914) [3] that there are very many zeros at $Re(s) = 1/2$ and these will appear as poles in $F(s)$. This proves the Claim.QED.

11 APPENDIX IV: The sequence of λ's in L(N), are equivalent to Coin Tosses

In this paper we showed in Theorem 3, that the $\lambda(n)$ have an exactly equal probability of being $+1$ or -1. Then in Appendix III, we showed that the sequence $\lambda(1), \lambda(2), \lambda(3), ..., \lambda(n), ...$ can never be cyclic. The latter result in the minds of most computer scientists would be interpreted as that the sequence of λ's by virtue of it being non-repetitive, is truly random,(Knuth (1968); Press etal (1986)) and hence it is legitimate to treat the sequence as a result of coin tosses and thus one can then say that $L(N) = \Sigma_1^N \lambda(n)$, will tend to $\sqrt{N}$ thus proving RH, by using the arguments given at Section 5.

However, this done, there would be some mathematicians who may remain unconvinced, because we have not strictly proved that the λ's in the series are independent. The purpose of this Appendix [‖] is to prove that this is indeed the case. This allows us to demonstrate the $\lambda-$sequence has the same properties as, and is statistically equivalent to, coin tosses, thus placing our proof of RH beyond any doubt.

We again consider the series $L(N) = \sum_{n=1}^{N} \lambda(n)$, which is re-written as:

$$L(N) = \sum_{n=1}^{N} X_n \tag{1}$$

It has already been proved in this paper that, over the set of all positive integers, the respective probabilities that an integer n has an odd or even number of prime factors are equal. So, X_n $(= \lambda(n))$, can with equal probability, be either +1 or -1. It will now be shown that the values of X_i and X_j, $i \neq j$, are independent of each other, as $n \to \infty$, and so will become the equivalent of ideal-coin tosses.

11.1 The λ values as a deterministic series

We first show that the λ's in the natural sequence, far from being random, are actually perfectly predictable and therefore deterministic. That is, knowing the λ's (and the primes) up to N, we can directly obtain (without resorting to factorisation) the λ's (and primes) up to $2N$ thus:

We obtain integers m in the range $N < m \leq 2N$ by multiplying the integers n and q in the range $1 < n, q \leq N$, such that $N < nq \leq 2N$ and then using the property $\lambda(q * n) = \lambda(q) * \lambda(n)$ to find $\lambda(m = q * n)$. However, not all the numbers in the range $N < m \leq 2N$ will be covered by such multiplications. That is, there will be 'gaps' in the natural sequence left in the aforesaid multiplications, where no n and q can be found for some m's in $N < m \leq 2N$. These m's will identified as prime numbers. The λ of a prime is -1. Thus, by

‖ I thank my brother Vinayak Eswaran for providing the kernel of the proof given in this section.

knowing the λ's and the primes up to N, we can predict the λ's (and primes) up to $2N$. This process can be repeated *ad-infinitum* to compute the λ's of the natural sequence up to any N, from just $\lambda(1)$=1 and $\lambda(2)$=-1.

We emphasize that any other method of evaluating the λ's, including direct factorisation, must perforce yield the same sequence as the method above. Therefore, this method offers a complete description of the determinism embedded in the series.

11.2 Relationships and dependence between λ's

We note that every integer n has a direct relationship (which we will call a d-relationship) with all numbers $n * p$, where p is any prime number. We can define higher-order d-relatives in the following way: the integers (n, $n * p$) are in a first-order d-relationship, (n, $n * p * q$) are in a second-order one, and (n, $n * p * q * r$) are in a third-order one, and so on, where p, q, r are primes (not necessarily unequal).

In the deterministic generation of λ's outlined above, it is clear that their values will be determined through d-relationships, which would thereby make their respective values dependent on each other. It is evident that the λs of two d-relatives n and $m(> n)$, are *dependent* on each other and that $\lambda(m) = (-1)^o \lambda(n)$, where o is the order of the relationship.

There is another kind of relationship we must also consider: we can have a c- (or *consanguineous*) relationship between two non-d-related integers m and n if they are both d-relatives of a common ('ancestor') integer smaller than either of them. So we can trace back the λ's along one branch to the common ancestor and trace it up the other to find the λ of the other integer. It is convenient to take the common ancestor as the largest possible one, which would be the greatest common factor of the two integers, which we shall call G.

Now we ask the question, when are m and n not related? When they have neither a d-relationship nor a c-relationship with each other. That is, when they are co-primes: as then neither integer would appear in the sequence of multiplications that produce the other by the deterministic iterative method. In such a situation, *the λ of neither is dependent on the other, so their mutual λ's are independent.*

Now consider two c-relatives, m and n, which share the greatest common factor G. We can write $m = G * P$ and n=$G * Q$, where P and Q are chosen appropriately. As G is the greatest common factor of m and n, it is clear that P and Q are co-primes. Now we consider the relationship between $\lambda(m)$ and $\lambda(n)$ and explore their relatedness. This turns out to be self-evident: As $\lambda(m)$ =$\lambda(G) * \lambda(P)$ and $\lambda(n)$ =$\lambda(G) * \lambda(Q)$, and we know that $\lambda(P)$ and $\lambda(Q)$ are independent of each other, it follows that $\lambda(m)$ and $\lambda(n)$ are also independent of each other.[**]

11.3 The unpredictability of λ values from a finite-length sequence: d-relatives

We have concluded above that the only λ's in $L(N)$ that are dependent are those between d-relatives, where the smaller integer is a factor of the other. We see that the distance of two such "first-order" relatives, n and $n * p$, from each other is $n(p - 1)$ which increases without bound with n. Further all the first-order d-relatives of n also have relative distances with each other that are at least as great as n (as their respective p's will differ at least by 1). Thus the d-relationship between numbers is a web with increasing distances between their first-order relatives[††]. It is also easy to see that the higher-order d-relatives of any integer n will also be at a distance of at least n from n itself and from each other.

Now we consider if we would be able to predict $\lambda(N + 1)$ if we know *only* the λ's between $N - L < n \leq N$, where L is some finite number? We would be able to do so *only* if $N + 1$ is a d-relative (of any order) of any of the numbers $N - L < n \leq N$. However, for n large enough the d-relatives of $N + 1$ will be far from it and would not come in the range of numbers $N - L < n \leq N$. So essentially, there is no way of predicting $\lambda(N + 1)$ from the range of L λ's coming before it. This means $\lambda(N + 1)$ is independent of the range of L λ's coming before it. Therefore, the λ's on all finite lengths are independent of each other, as $N \to \infty$.

11.4 Closure

We have investigated the dependence of λ's appearing in $L(N)$ in the natural sequence $n = 1, 2, 3, .., $. We first show that the λ's are in a perfectly deterministic sequence (which is not random in the slightest way, except in the unpredictable discovery of primes) that allows us to obtain all of them up to any integer N by knowing

[**]It may be noticed that m and n belong to different towers. It is worth mentioning that the arguments made here in Appendix IV, can be couched in the language of towers as we did in Sections 2 and 3.

[††]How rapidly the relationship distance increases can be gauged from the fact that the 2^r sequence, which has the *slowest* increases, nevertheless will have its 100^{th} element placed at around $n \approx 10^{30}$ in the natural sequence, and the distance to the 101^{st} element will also be 10^{30}!

only that $\lambda(1) = 1$, $\lambda(2) = -1$, $\lambda(q * n) = \lambda(q) * \lambda(n)$, and that $\lambda(p) = -1$ for any prime p. We then propose that the λ's of two integers m and n can be dependent only if the integers are connected through the sequence of multiplications involved in the deterministic process. If they are not so related, as would happen if they are co-primes, their λ's would be independent. We then investigate the only two possible types of relationships and show that one, the d-relationship, leads to dependencies between numbers that are increasingly distant. The other, the c-relationship, is shown to give independent λ's. The result obtained is that that the λ's in any finite sequence are independent, as $N \to \infty$. QED

12 APPENDIX V

An Arithmetical Proof for $|L(N)| \sim N^{1/2}$ as $N \to \infty$

In this appendix we provide an alternate, but this time an arithmetical, proof of the asymptotic behavior of the summatory Liouville function, viz.$|L(N)| = \sqrt{N}$ as $N \to \infty$ However in order to do this we first require to prove a theorem on the number of distinct prime products in the factorization of a sequence of integers and their exponents. We will be considering special types of Sets $S^-(N)$ and $S^+(N)$ which contain a sequence of consecutive integers, they are defined below; each are of length $\sqrt{N}$, where in this section N will always be a perfect square. A collection of all such sets will contain as its members all the integers and the intersection of any two different sets will be null. See Tables 1.1 to 1.4 in Appendix VI. We will be studying the contribution of such sets to the summatory function $L(N)$ in order to determine the asymptotic behavior of the latter as $N \to \infty$

Theorem A5: Consider the sequence S comprising $M(N)$ consecutive positive integers, defined by $S^-(N) = \{N - M(N) + 1, N - M(N) + 2, N - M(N) + 3, ..., N\}$, where $M(N) = \sqrt{N}$. Then every number in $S^-(N)$ will firstly belong to different towers,* and further every number will: (a) differ in its prime factorization from that of any other number in $S^-(N)$ by at least one distinct prime† OR (b) in their exponents.

The statement of this theorem can be roughly considered as an extremely weak form of Grimm's conjecture,(1969) which states that a sequence of k consecutive composite integers will have at least k distinct primes in their factorization, also see Ramachandra etal.(1975), Grimm's theorem though not proved, yet, has been verified for very many subsets, see: S. Laishram and Ram Murty (2006,2012), and Balasubramanian, etal [2009]. We do not need this very strong version for our arguments.

We first take up the task to prove (a) because it is by far the more common occurrence. In case condition (a) does not hold in a particular situation then condition (b) is always true, because of the uniqueness of factorization.

Proof:

Let there be k primes in the sequence $S^-(N)$. Denote the j integers in the sequence that are not primes by the products $p_i b_i$, $i = 1, 2..., j$, where p_i is a prime and, obviously, $k + j = M(N)$. Denote the subset of these non-prime integers by J. There is no loss of generality if we assume the primes p_i in the products $p_i b_i$, $i = 1, 2..., j$, to be less than $\sqrt{N} - 1/2$ and also the smallest of prime in the product.‡

To prove the theorem, we compare two arbitrary members, $p_i b_i$ and $p_j b_j$, $i \neq j$, belonging to set J.

Case 1: Suppose $p_i \neq p_j$. If $b_i \neq b_j$, b_i must contain a prime that does not appear in the factorization of b_j (and hence $p_i b_i$ must be different from $p_j b_j$ by this prime). For if b_i and b_j do not differ by a prime, we must have $b_i = b_j \equiv b$. This means the difference of $p_i b_i - p_j b_j = (p_i - p_j)b$ is larger than $\sqrt{N}$ in absolute value. This is not possible since the members of the sequence $S^-(N)$ cannot differ by more than $\sqrt{N}$. Therefore b_i must differ from b_j by a prime in its factorization.(One may think that it may be plausible that $b_i = b^r$ and $b_j = b^m$, where r and m are positive integers, in which case $p_i b_i$ differs from $p_j b_j$ only in the prime p_i. However, this eventuality will never arise because then the difference between $p_i b_i$ and $p_j b_j$ will be more than $\sqrt{N}$.)

Case 2: Suppose $p_i = p_j \equiv p$ then b_i and b_j must differ by a prime factor or their exponents are different. Because of 'unique factorization', if they do not differ by a prime factor it means $p_i.b_i = p_j.b_j = p.b$, unless the factors of b_i and b_j are of the form: $b_i = p_1^{r_1}.p_2^{r_2}...p_k^{r_k}$ and $b_j = p_1^{r'_1}.p_2^{r'_2}...p_k^{r'_k}$ which implies that is $r_l = r'_l$ is not

*Two numbers $n = m.p^\alpha.u$ and $n' = m.p^\beta.u$, $(n < n')$, of the same tower, cannot both belong to the set $S^-(N)$ because they will be too far separated to be within the set, as their ratio $n'/n \geq p \geq 2$

†For example, if two numbers c and d in S are factorized as $c = p_1^{e_1} p_2^{e_2}$ and $d = p_3^{e_3} p_4^{e_4}$ then at least one of the primes p_3 or p_4 will be different from p_1 or p_2.

‡This is readily seen as follows. Since every member of J lies between $N - \sqrt{N}$ and N, clearly any composite member, written as a product ab, cannot have both integers a and b less than $\sqrt{N} - 1/2$. (We are invoking the fact that $\sqrt{(N - \sqrt{N})} = \sqrt{N} - 1/2$, approximately.) Let a be the smaller of the two numbers, and so $a < \sqrt{N} - \frac{1}{2}$ and $b > \sqrt{N} - \frac{1}{2}$. If a is a prime number, set $p = a$. If a is not a prime number, factorize it and pick the smallest prime p which is one of its prime factors.

true for all r_l, $l = 1, 2..k$, hence in this case the exponents are different (actually this case is very rare. It can be shown: the case $k = 2$ cannot occur and therefore if at all this case occurs, k must be greater than 3).

Since $p_i b_i$ and $p_j b_j$ are arbitrary members of the set J, it follows that every integer in J must differ from another integer in J by at least one prime in its factorization or by its exponent, thus making the $\lambda-$values of any two members of the set $S^-(N)$ not dependent on each other . □

The above theorem has profound implications for the λ-values of the numbers in the sequence $S^-(N)$. If we take the primes to occur randomly (or at least pseudo-randomly), the λ-value of each of these $M(N)$ integers—although deterministic and strictly determined by the number of primes in its factorization—cannot be predicted by the λ-value of any other number in the sequence $S^-(N)$. That is, the λ-value of any number in $S^-(N)$ can be considered to be statistically independent of the λ-value of another member of this sequence, primarily because they stem from different towers and also because (as we have proved in A5) any two such numbers differ by at least one prime. Hence the λ-values in the sequence $S^-_\lambda \equiv \{\lambda(N - M(N) + 1), \lambda(N - M(N) + 2), \lambda(N - M(N) + 3), ..., \lambda(N)\}$, in which each member has a value either $+1$ or -1, would appear randomly and be statistically similar. By this we also deduce that two different sequences of $\lambda-$values defined on two different sets (say) $S^-(N)$ and $S^-(N')$ with $N \neq N'$ are statistically similar, because they have the same properties which also means that they can be separately compared with other sequences of coin tosses and the comparison should yield statistically similar results.

We will use these deductions to obtain the main result of this appendix viz $a = 1/2$ in the expression $|L(N)| = N^a$ as $N \to \infty$

Although it is not explicitly required for what follows, we note that it is not hard to prove that the sequence $S^+(N) \equiv \{N + 1, N + 2, N + 3, ..., N + M(N)\}$ of length $M(N)$ also behaves similarly. That is, every member of $S^+(N)$ satisfy condition (a) OR (b) of the above Theorem for $S^-(N)$ stated above. The proof mimics the one provided above and so is omitted.[§] Hence the λ-values in the sequence $S^+_\lambda \equiv \{\lambda(N + 1), \lambda(N + 2), \lambda(N + 3), ..., \lambda(N + M(N))\}$, in which each member has a value either $+1$ or -1, would also appear randomly and behave statistically similarly.

12.1 Arithmetical proof of $|L(N)| \sim \sqrt{N}$, as $N \to \infty$

We now show that if the summatory Liouville function

$$L(N) = \sum_{n=1}^{N} \lambda(n), \tag{1}$$

takes the asymptotic form

$$|L(N)| = C\, N^a, \tag{2}$$

where C is a constant, then we must have:

$$a = 1/2. \tag{3}$$

Throughout this subsection we will always assume that N is a very large integer.

Consider the sequence of consecutive integers of length $M(N) = \sqrt{N}$:

$$S_N = \{N - \sqrt{N} + 1, N - \sqrt{N} + 2, N - \sqrt{N} + 3, \cdots\cdots, N\} \tag{4}$$

Each of the $M(N)$ integers in the sequence S_N can be factorized term by term and would differ from another member in S_N by at least one prime or exponent,(as proved in the above theorem).[¶]

Now since, N is large, all the primes involved may be considered random numbers (or pseudo-random numbers), therefore as reasoned above, we can conclude that the $\lambda-$sequence associated with S_N viz.

$$\{\lambda(N - \sqrt{N} + 1), \lambda(N - \sqrt{N} + 2), \lambda(N - \sqrt{N} + 3), \cdots\cdots, \lambda(N)\} \tag{5}$$

will take values which are random e.g.

$$\{-1, +1, +1, , -1, +1, \cdots, +1\} \tag{6}$$

where in the above example $\lambda(N - \sqrt{N} + 1) = -1, \lambda(N - \sqrt{N} + 2) = +1$ etc. Furthermore, since the λ-values have an equal probability of being equal to $+1$ or -1 (Theorem 3) and the sequence is non-cyclic (Theorem 11.1,

[§]This implies, interestingly, that by choosing N to be consecutive perfect squares, the entire set of positive integers can be envisaged as a union of mutually exclusive sequences like $S^-(N)$ and $S^+(N)$.

[¶]Therefore, in the terminology of Sections 2 and 3, each of them will mostly belong to different Towers.

in Appendix 3), the above sequence will have the statistical distribution of a sequence of tosses of a coin (Head = +1,Tail = −1). But we already know from Chandrasekhar(1943) that if the λ's behave like coin tosses then $|L(N)| \sim \sqrt{N}$, as $N \to \infty$. However, we do not know whether the entire sequence of λ's occurring in Eq.(13.1) behaves like coin tosses; for any given N, it is only the subsequence $\{\lambda(N-\sqrt{N}+1), \lambda(N-\sqrt{N}+2), \lambda(N-\sqrt{N}+3), \cdots\cdots, \lambda(N)\}$ of length $M(N) = \sqrt{N}$ that does behave like coin tosses.

On the other hand if we had a sequence of length N, of real coin tosses (say) $c(n), n = 1, 2....N$, where $c(n) = \pm 1$, then the cumulative sum, $L_c(N)$, of the first N of such coin tosses is given by:

$$L_c(N) = \sum_{n=1}^{N} c(n). \tag{7}$$

Then for N large we do know from Chandrashekar (1943) that

$$|L_c(N)| \sim \sqrt{N}. \tag{8}$$

We can then estimate the contribution $P_{1/2}$ to $L_c(N)$ from the last $M(N) = \sqrt{N}$ terms in Eq.(13.7), this would be:

$$\begin{aligned} P_{1/2} &= \sum_{n=N-\sqrt{N}+1}^{N} c(n) \\ &= L_c(N) - L_c(N-\sqrt{N}) \end{aligned} \tag{9}$$

Now since Eq.(13.7) represents perfect tosses Eq. (13.9) becomes

$$P_{1/2} = \sqrt{N} - (N-\sqrt{N})^{1/2} = \frac{1}{2} - \frac{1}{8}\frac{1}{\sqrt{N}},$$

that is,

$$P_{1/2} = O(1) \tag{10}$$

In Eq. (13.10), $P_{1/2}$ is the contribution to $L_c(N)$ from the last $M(N) = \sqrt{N}$ tosses of a total of N tosses of a coin. We shall consider the value of $P_{1/2}$ as the benchmark with which to compare the contributions of the last $M(N) = \sqrt{N}$ terms of the summatory Liouville function.

Now coming back to the λ-sequence as depicted in the summation terms in Eq. (13.1), following Littlewood (1912) we shall suppose that the expression given in (13.2) is an ansatz‖ depicting the behavior of $L(N)$ for large N.

The task that we then set ourselves, is to estimate the value of the exponent a in the asymptotic behavior described in (13.2) $|L(N)| = C\,N^a$ which involves the $\lambda-$sequence. We do know that the λ-sequence does not all behave like coin tosses, but we have shown that there exist subsequences of λ's that exhibit a close correspondence to the statistical distribution of coin tosses and though such subsequences are of relatively short lengths $M(N)$, there are very many in number. Now a 'True' value of the exponent 'a' should be able to capture the correct statistics in all such subsequences and predict the behavior of coin tosses for such subsequences. We now investigate if such a True value for a exists and, if so, what its value should be.

We will estimate the contribution to $L(N)$ for the same subsequence (5) of length $M(N) = \sqrt{N}$, then the P when recomputed with an exponent $a \neq 1/2$ would give P_a:

$$P_a = \sum_{n=N-\sqrt{N}+1}^{N} \lambda(n) \qquad (13.5')$$

That is

$$\begin{aligned} P_a &= L(N) - L(N-\sqrt{N}) \\ &= CN^a - C(N-\sqrt{N})^a \end{aligned} \tag{11}$$

Simplifying by using Binomial expansion we have:

$$P_a = CaN^{a-\frac{1}{2}} - C\frac{a(a-1)}{1.2}N^{a-1} \tag{12}$$

‖Eq. (13.2) can be thought of as the first term in the asymptotic expansion of $L(N)$ for large N i.e. $|L(N)| = N^a(C + \frac{C_1}{N} + \frac{C_2}{N^2} + ...)$

From the properties of the λ's deduced from earlier results in this paper (Theorem 3, Appendices 3,4 and Theorem A5, page 21), we now know that in actuality the particular subsequence in Eq.(13.5) and Eq.(13.5′) contain random values of $+1$ and -1 and since the subsequence of λ's have the same statistics as those of coin tosses, P_a must be similar to $P_{1/2}$. Thus from (13.10) and (13.12)

$$P_a = O(1). \tag{13}$$

From (13.11) this means that

$$CaN^{a-\frac{1}{2}} = O(1) \tag{14}$$

Since N is arbitrary and very large, this is impossible unless the condition

$$a = \frac{1}{2} \tag{15}$$

strictly holds.**

Hence we have proved $a = 1/2$. Since for consistency†† , condition (15), which arises from (13), is mandatory and therefore $|L(N)| \sim \sqrt{N}$ describes the asymptotic behavior of the summatory Liouville function. □

13 APPENDIX VI

On Coin Tosses and the Proof of Riemann Hypothesis

This Appendix has been written in such a manner that it can be read as a supplement to the main paper and the first five appendices.

In the main part of this paper and the forgoing appendices, which we denote as: [MP and A's], we had proved the validity of the Riemann Hypothesis (RH). In this Appendix (VI), we perform a numerical analysis and provide supporting empirical evidence that is consistent with the formal theorems that were key to establishing the correctness of the RH. In particular, the numerical results of the statistical tests performed here are firmly consistent with the proposition (formally proved in the paper cited above) that the values taken on by the Liouville function over large sequences of consecutive integers are random. By performing this exhaustive numerical analysis and statistical study we feel that we have provided a clearer understanding of the Riemann Hypothesis and its proof.

1. Introduction

The Riemann zeta function, $\zeta(s)$, is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{1}$$

where n is a positive integer and s is a complex variable, with the series being convergent for $Re(s) > 1$. This function has zeros (referred to as the trivial zeros) at the negative even integers $-2, -4, \ldots$. It has been shown‡‡ that there are an infinite number of non-trivial zeros on the critical line at $Re(s) = 1/2$. Riemann's Hypothesis (RH), which has long remained unproven, claims that all the nontrivial zeros of the zeta function lie on the critical line. The main paper contains the proof [MP and A's]

In this technical note, we provide a more concrete understanding and appreciation of the steps involved in the proof of the Riemann Hypothesis by supplying supporting empirical evidence for those various theorems which were proved and which had played a key role in the proof of the RH. In what follows we first give a brief summary of how the RH was proved in [MP and A's]. The proof followed the primary idea that if the zeta function has zeros only the critical line, then the function $F(s) \equiv \zeta(2s)/\zeta(s)$ cannot be analytically continued to the left from the region $Re(s) > 1$, where it is analytic, to the left of $Re(s) < 1/2$. This point was recognized by Littlewood as far back as 1912.* The function $F(s)$ can be expressed as (see Titchmarsh (1951, Ch. 1)):

**In the above we tacitly assumed that $a > 1/2$, but $a < 1/2$ is not possible because then P_a will become zero. This implies that $dL/dN = 0$, meaning $|L(N)|$ will be a constant. But this again is impossible from Theorem 1, which would imply that $F(s)$ can be analytically continued to $Re(s) = 0$—an impossibility because of the presence of an infinity of zeros at $Re(s) = 1/2$, first discovered by Hardy.

††It may be noted that for every (large) N there is a set S_N, Eq (13.4), containing $M = \sqrt{N}$ consecutive integers whose λ-values behave like coin tosses; but there are an infinite number of integers N and therefore there are an infinite number of sets S_N, for which (13) must be satisfied.

‡‡This was first proved by Hardy (1914).

*It may be noted that Littlewood studied the function $1/\zeta(s)$ whereas we, in our analysis study $F(s) \equiv \zeta(2s)/\zeta(s)$. This has made things simpler.

$$F(s) = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}, \tag{2}$$

where $\lambda(n)$ is the Liouville function defined by $\lambda(n) = (-1)^{\omega(n)}$, with $\omega(n)$ being the total number of prime numbers in the factorization of n, including the multiplicity of the primes. The proof of RH in [MP and A's] requires also the summatory Liouville function, $L(N)$, which is defined as:

$$L(N) = \sum_{n=1}^{N} \lambda(n) \tag{3}$$

The proof crucially depends on showing that the function $F(s) = \zeta(2s)/\zeta(s)$, has poles only on the critical line $s = 1/2 + i\sigma$, which translates to zeros of $\zeta(s)$, on the self same critical line $s = 1/2 + i\sigma$, because all the values of s which appear as poles of $F(s)$ are actually zeros of $\zeta(s)$, except for $s = 1/2$. Since, the trivial zeros of $\zeta(s)$ which occur at $s = -2, -4, -6....$ that is negative even integers, conveniently cancel out from numerator and denominator of the expression in $F(s)$), leaving only the non trivial zeros, also the pole of $\zeta(2s)$ will appear as a pole of $F(s)$, at $s = 1/2$. So it just remains to show that all the poles of $F(s)$ lie on the critical line. This was the Primary task of the paper.

The crucial condition then is that $F(s)$ is not continuable to the left of $Re(s) < 1/2$, and therefore that the zeta function have zeros only on the critical line,† is that the asymptotic limit of the summatory Liouville function be $|L(N)| \sim C\, N^{1/2}$, where C is a constant. Therefore, to provide a rigorous proof of the validity of the Riemann Hypothesis, [MP and A's] investigated the asymptotic limit of $L(N)$. The work involved the establishment of several relevant theorems, which were then invoked to eventually prove the RH to be correct.

We now state some of these important theorems‡).

***Theorem* 1:**

In the set of all positive integers, for every integer which has an even number of primes in its factorization there is another unique integer (its twin) which has an odd number of primes in its factorization.

Remark: Theorem 1 gives us the formal result that $Pr(\lambda(n) = +1) = Pr(\lambda(n) = -1) = 1/2$, where Pr denotes probability. That is, the λ-function behaves like an 'ideal coin'.

***Theorem* 2:**

Consider the sequence $S_-(N)$ comprising $\mu(N)$ consecutive positive integers, defined by $S_-(N) = \{N - \mu(N) + 1, N - \mu(N) + 2, N - \mu(N) + 3, ..., N\}$, where $\mu(N) = \sqrt{N}$. Then every number in $S_-(N)$ will differ in its prime factorization from that of every other number in $S_-(N)$ by at least one distinct prime.§

Remark: It is not hard to prove that the sequence $S_+(N) \equiv \{N + 1, N + 2, N + 3, ..., N + \mu(N)\}$ of length $\mu(N)$ also behaves similarly. That is, every member of $S_+(N)$ differs from every other member by at least one prime in its factorization. This implies, interestingly, that by choosing N to be consecutive perfect squares, the entire set of positive integers can be envisaged as a union of mutually exclusive sequences like $S_-(N)$ and $S_+(N)$.

It follows that the λ-values in the sequences $S_-^{\lambda}(N) \equiv \{\lambda(N - \mu(N) + 1), \lambda(N - \mu(N) + 2), ..., \lambda(N)\}$ and $S_+^{\lambda}(N) \equiv \{\lambda(N + 1), \lambda(N + 2), \lambda(N + 3), ..., \lambda(N + \mu(N))\}$, in which each member has a value either $+1$ or -1, would also appear randomly and be statistically similar to sequences of coin tosses.

Since the number of members in the sequences $S_-(N)$, $S_-^{\lambda}(N)$, $S_+^{\lambda}(N)$, and $S_+^{\lambda}(N)$ is given by $\mu(N) = \sqrt{N} \to \infty$ as $N \to \infty$, the behavior of the λ-values of very large integers should coincide with that of a sequence of coin tosses. This intuition was formally confirmed in Appendix V.

***Theorem* 3:**

The summatory Liouville function takes the asymptotic form $|L(N)| = C\, N^{1/2}$,C is a constant. It can be shown that $C = \sqrt{\frac{2}{\pi}}$. It may be mentioned here that Littlewood's condition is fairly tolerant: As long as asymptotically, for large N, $|L(N)| = C\, N^{1/2}$, and C is any finite constant, R.H. follows. This 'tolerance' is reflected in the value of χ^2 (below) as may be deduced, after a study of the following.

Remark: The form of the summatory Liouville function in Theorem 3 is precisely what we would expect for a sequence of unbiased coin tosses. This, along with a sufficient condition derived by Littlewood (1912), shows

†Riemann had already shown that symmetry conditions ensure that there will be no zeros $0 < Re(s) < 1/2$ if it is found that there are no zeros in the region $1/2 < Re(s) < 1$

‡In addition to the theorems given below, a necessary theorem which states that: The sequence $\lambda(1), \lambda(2), \lambda(3), ..., \lambda(n), ...,$ is asymptotically non-cyclic, (i.e. it will never repeat), was also proved, in [MP and A's], the theorems are numbered differently

§For example, if two numbers c and d in S are factorized as $c = p_1^{e1} p_2^{e2}$ and $d = p_3^{e3} p_4^{e4}$ then at least one of the primes p_3 or p_4 will be different from p_1 or p_2.

that $F(s)$ is analytic for $Re(s) > 1/2$ and $Re(s) < 1/2$, thereby leaving the only possibility that the non-trivial zeros of $\zeta(s)$ can occur only on the critical line $Re(s) = 1/2$.

In the following sections, by comparing the λ-sequences obtained for large sets of consecutive integers with (binomial) sequences of coin tosses, we show that the statistical distributions of the two sets of sequences are consistent with the claims of the above theorems. To this end, we apply Pearson's 'Goodness of Fit' χ^2 test. The software program *Mathematica* developed by Wolfram has been used in this technical report to aid in the prime factorization of the large numbers that this exercise entails.

The compelling bottom line that emerges from this empirical study is that it is extremely unlikely, in fact statistically impossible, that for large N, the sequences of λ-values can differ from sequences of coin tosses. It is this behavior of the Liouville function, recall, that delivers Theorem 3 above. And this Theorem, in turn, nails down all the non-trivial zeros of the zeta function to the critical line [MP and A's].

2. χ^2 Fit of a λ-Sequence

In this section we will derive an expression of how closely a λ sequence corresponds to a binomial sequence (coin tosses). We follow the exposition given in Knuth (1968, Vol. 2, Ch. 3); and then derive a very important expression for a χ^2 fit of a λ-Sequence, given by Eq.(14.9) below.

Suppose we are given a sequence, $T(N_0, N)$, of N consecutive integers starting from N_0:
$T(N_0, N) = \{N_0,\ N_0 + 1,\ N_0 + 2,\ N_0 + 3, ..., N_0 + N - 1\}$
and the sequence, $\Lambda(N_0, N)$, of the corresponding λ-values:
$\Lambda(N_0, N) = \{\lambda(N_0),\ \lambda(N_0 + 1),\ \lambda(N_0 + 2),\ \lambda(N_0 + 3), .. ,\ \lambda(N_0 + N - 1)\}$.

We ask how close in a statistical sense the sequence $\Lambda(N_0, N)$ is to a sequence of coin tosses or, in other words, a binomial sequence. By identifying $\lambda(n) = 1$ as *Head* and $\lambda(n) = -1$ as *Tail*, for the n^{th} 'toss', we may perform this comparison. If this is really the case then statistically $\Lambda(N_0, N)$ should resemble a binomial distribution, we can then compute the χ^2 statistic as follows.

$$\chi^2(N) = \frac{(P - E_P)^2}{E_P} + \frac{(M - E_M)^2}{E_M}, \tag{4}$$

where P and M are the actual number of +1s (*Heads*) and −1s (*Tails*), respectively, in the $\Lambda(N_0, N)$ sequence, E_P and E_M are the expectations of the number of +1s and −1s in the probabilistic sense. From Theorem 1 it immediately follows that, for large N,

$$E_P = E_M = N/2. \tag{5}$$

We define $L(N_0, N)$ as the additional contribution to the summatory Liouville function of N consecutive integers starting from N_0:

$$L(N_0, N) = \sum_{n=N_0}^{N_0+N-1} \lambda(n). \tag{6}$$

For brevity we will denote $\widehat{L} \equiv L(N_0, N)$ and since (6) contains P terms which are equal to +1s and M terms which are equal to −1s, we can write:

$$P - M = \widehat{L}, \tag{7}$$

and

$$P + M = N. \tag{8}$$

Using (7)and (8) we see that $P = (N + \widehat{L})/2$ and $M = (N - \widehat{L})/2$ and from (5) we deduce $P - E_P = \widehat{L}/2$ and $M - E_M = -\widehat{L}/2$ and thus equation (4) gives us the very important χ^2 relation which is satisfied by every $\Lambda(N_0, N)$ sequence involving the factorization of N consecutive integers starting from N_0:

$$\chi^2(N) = \frac{[L(N_0, N)]^2}{N}. \tag{9}$$

Note that the it was possible to derive an expression for χ^2 for large$N$ only because of Theorems 1, 2, and 3. Now we particularly choose N to be the square of an integer and the sequence of length $\mu(N) = \sqrt{N}$ starting from the integer $N_0 = N - \sqrt{N} + 1$ and then taking the $\sqrt{N}$ consecutive terms of the λ-sequence, we obtain

$$\Lambda(N_0, \sqrt{N}) = \{\lambda(N - \sqrt{N} + 1),\ \lambda(N - \sqrt{N} + 2),\ \lambda(N - \sqrt{N} + 3),\ ..., \lambda(N)\ \}, \tag{10}$$

and the corresponding $\chi^2(\sqrt{N})$ for such a sequence (which is of length $\sqrt{N}$) can be obtained from Theorem 3 and (9) as

$$\begin{aligned}\chi^2(\sqrt{N}) &= \frac{[C\sqrt{\sqrt{N}}]^2}{\sqrt{N}} \\ &= C^2. \end{aligned} \tag{11}$$

Equation (11) of course, should be interpreted as the average value of a sequence such as $\Lambda(N_0, \sqrt{N})$ of length $\sqrt{N}$ given in the expression (10). In this report we perform the χ^2 'Goodness of Fit' tests for very many sequences of the type $\Lambda(N_0, \sqrt{N})$ with varying lengths and very large values of N to examine whether these sequences are statistically indistinguishable from coin tosses. In this manner, we provide empirical support for the claims of the theorems formally proved in [MP and A's] and, therefore, for the proof of the Riemann Hypothesis.

3. Numerical Analysis of Sequence $\Lambda(N_0,\ \sqrt{N})$ and its χ^2 Fit

In this section, we consider sequences of length $\sqrt{N}$, starting from $N_0 = N - \sqrt{N} + 1$ or $N + 1$ where N is a perfect square. We use *Mathematica* to compute $L(N_0, N)$.¶

In the table below we list the sequences in the following format. We define the sequences:

$$S_-(N) = \{N - \sqrt{N} + 1,\ N - \sqrt{N} + 2,\ ..., N\}, \tag{12}$$

$$S_+(N) = \{N + 1,\ N + 2,\ ...,\ N + \sqrt{N}\}, \tag{13}$$

and the partial sums of the λs of the two sequences defined above are defined by the expressions:

$$L(S_-) \equiv L(N - \sqrt{N} + 1, N) = \lambda(N - \sqrt{N} + 1) + \lambda(N - \sqrt{N} + 2) + ... + \lambda(N), \tag{14}$$

$$L(S_+) \equiv L(N + 1, N + \sqrt{N}) = \lambda(N + 1) + \lambda(N + 2) + ... + \lambda(N + \sqrt{N}). \tag{15}$$

The formal proof of the Riemann Hypothesis in [MP and A's] proceeded as follows. The sequences $\Lambda(N - \sqrt{N} + 1, \sqrt{N})$ and $\Lambda(N + 1, \sqrt{N})$ were shown to behave like coin tosses for every N (large) over sequences of length $\sqrt{N}$, where N is taken to be a perfect square. On taking N to be consecutive perfect squares, the lengths of the consecutive sequences naturally increase. Using this procedure, we obtain sequences that can span the *entire set* of positive integers (consult the first five columns of Tables 1.1 to 1.4). Since the λs within each segment behave like coin tosses, from the work of Chandrashekar (1943) it follows that the summatory Liouville function $L(N)$ must behave like $C\sqrt{N}$ as $N \to \infty$. The validity of RH follows, by Littlewood's Theorem, from the fact that $F(s)$ cannot then be continued to the left of the critical line $Re(s) = 1/2$ because of the appearance of poles in $F(s)$ on the line, each pole corresponding to a zero of the zeta function $\zeta(s)$.

Statistical Tests

We shall now test the following null hypothesis H_0 against the alternative hypothesis H_1 in the following generic forms:

H_0: The sequence $\Lambda(N_0, N)$ has the same statistical distribution as a corresponding sequence of coin tosses (i.e. binomial distribution with $Prob(H) = Prob(T) = 1/2$).

H_1: The sequence $\Lambda(N_0, N)$ has a different statistical distribution than a corresponding sequence of coin tosses (i.e. binomial distribution with $Prob(H) = Prob(T) = 1/2$).

The critical value for chi square is $\chi^2_{crit} = 3.84$, for the standard 0.05 level of significance. (In our case, the relevant degrees of freedom equal to 1.) Assuming that H_0 is true, if chi square is less than χ^2_{crit} the null hypothesis is accepted.

It should be noted that the tests conducted here are not merely exploratory statistical exercises to discern possible patterns in the λ-sequences. Rather, the tests here are informed by theory. We have formally shown in [MP and A's] that, over the set of positive integers, the probability that λ takes on the value $+1$ or -1 with

¶A typical Mathematica command which calculates the expression $\sum_{n=J}^{K} \lambda(n)$ is:
Plus[LiouvilleLambda[Range[J,K]]].For instance, the command which sums
the $\lambda(n)$ from $n = 25,000,001$ to $25,005,000$ is:
Plus[LiouvilleLambda[Range[25000001, 25005000]]], which will give the answer $= -42$.

equal probability and that, over sequences that are increasing in N, the λ draws are random. Thus statistical evidence consistent with these claims merely bolster what has already been formally demonstrated.

The behavior of the $\Lambda(N_0, \sqrt{N})$ sequences are verified to be indeed like coin tosses for a very large number of cases and the results are summarized in the tables below. Let us take an example from Table 1.1. The third row gives the χ^2 result for the sequence of length 1001, starting from 1001001. We can factorize each of these numbers as:

$1001001 = 3 \times 333667$; $1001002 = 2 \times 500501$; $1001003 = prime$;
$1001004 = 2^2 \times 3 \times 83417$;.........., $1001999 = 41 \times 24439$;
$1002000 = 2^4 \times 3 \times 5^3 \times . 167$; $1002001 = 7^2 \times 11^2 \times 13^2$

and hence we can evaluate the corresponding λ-sequence, by using the definition $\lambda(n) = (-1)^{\omega(n)}$, with $\omega(n)$ being the total number of prime numbers (multilpicities included) in the factorization of n. We find that:

$\Lambda(1001001, 1001) = \{\lambda(1001001), \lambda(1001002), ..., \lambda(1002000), \lambda(1002001)\}$
$= \{1, 1, -1, 1, 1, 1, 1, -1, 1, -1, 1, ..., 1, -1, 1\}$.

The partial sum of all the 1001 λs shown in the sequence above adds up to 49. We then estimate how close the sequence $\Lambda(1001001, 1001)$ is to a Binomial distribution, i.e. of 1001 consecutive coin tosses. The observed value $\chi^2 = 2.4$ for this sequence of λs is well below the critical value $\chi^2_{crit} = 3.84$ (for a one degree of freedom) at the standard significance level of 0.05. Thus the sequence $\Lambda(1001001, 1001)$ is statistically indistinguishable from a Binomial distribution obtained by 1001 consecutive coin tosses if we consider $Head = +1$ and $Tail = -1$. In fact, it so happens that out of the 10 sequences shown in Table 1.1 this chosen example has the largest value of χ^2; the other sequences have a much lower χ^2 value and the average value is 0.653 which hovers around the predicted average $C^2 = \frac{2}{\pi} = 0.637$. We see that the null hypothesis would be accepted even if the significance level were at 0.10, for which $\chi^2_{crit} = 2.71$.

We have calculated the χ^2 for larger and larger sequences see Tables1.2, Tables 1.3 and Tables 1.4 for even very large numbers $\sim 10^{10}$ and sequences involving 10^5consecutive integers in each case the sequences $\Lambda(N_0, \sqrt{N})$ behave like coin tosses thus lending emphatic empirical support consistent with the Theorems proved in [MP and A's],.

TABLE 1.1 Sequence of Consecutive Integers of Type $S_-(N)$ and $S_+(N)$ of Length 1000

No	*Type* of S(N)	$\sqrt{N}$	From	to	L(S)	χ^2
1.	S-	1000	999,001	1,000,000	6	0.036
2.	S_+	1000	1,000,001	1,001,000	10	0.100
3.	S-	1001	1,001,001	1,002,001	49	2.400
4.	S_+	1001	1,002,002	1,003,002	-37	1.368
5.	S-	1002	1,003,003	1,004,004	-12	0.144
6.	S_+	1002	1,004,005	1,005,006	-28	0.780
7.	S-	1003	1,005,007	1,006,009	3	0.009
8.	S_+	1003	1,006,010	1,007,012	-39	1.516
9.	S-	1004	1,007,013	1,008,016	12	0.143
10.	S_+	1004	1,008,017	1,009,020	6	0.036
	MEAN	χ^2 **FROM**	**999,001**	**1,009,020**	=	**0.653**

TABLE 1.2 Sequence of Consecutive Integers of Type $S_-(N)$ and $S_+(N)$ of Length 5000

No	*Type* of S(N)	$\sqrt{N}$	From	to	L(S)	χ^2
1.	S-	5000	24,995,001	25,000,000	0	0.0
2.	S_+	5000	25,000,001	25,005,000	-42	0.353
3.	S-	5001	25,005,001	25,010,001	-27	0.148
4.	S_+	5001	25,010,002	25,015,002	-103	2.12
5.	S-	5002	25,015,003	25,020,004	-76	1.155
6.	S_+	5002	25,020,005	25,025,006	48	0.461
7.	S-	5003	25,025,007	25,030,009	-13	0.034
8.	S_+	5003	25,030,010	25,035,012	119	2.831
9.	S-	5004	25,035,013	25,040,016	124	3.072
10.	S_+	5004	25,040,017	25,045,020	62	0.768
	MEAN	χ^2 **FROM**	**24,995,001**	**25,045,020**	=	**1.094**

TABLE 1.3 Sequence of Consecutive Integers of Type $S_{-}(N)$ and $S_{+}(N)$ of Length 10,000

No	*Type*	$\sqrt{N}$	From	to	L(S)	χ^2
1.	S_-	10000	99,990,001	100,000,000	-146	2.132
2.	S_+	10000	100,000,001	100,010,000	-88	0.774
3.	S_-	10001	100,010,001	100,020,001	-11	0.012
4.	S_+	10001	100,020,002	100,030,002	-43	0.185
5.	S_-	10002	100,030,003	100,040,004	8	0.064
6.	S_+	10002	100,040,005	100,050,006	36	0.130
7.	S_-	10003	100,050,007	100,060,009	23	0.053
8.	S_+	10003	100,060,010	100,070,012	-49	0.240
9.	S_-	10004	100,070,013	100,080,016	-20	0.040
10.	S_+	10004	100,080,017	100,090,020	112	1.254
	MEAN	χ^2 **FROM**	**99,990,001 TO**	**100,090,020**	=	**0.488**

TABLE 1.4 Sequence of Consecutive Integers of Type $S_{-}(N)$ and $S_{+}(N)$ of Length 100,000

No	*Type*	$\sqrt{N}$	From	to	L(S)	χ^2
1.	S_-	100,000	9,999,900,001	10,000,000,000	-232	0.538
2.	S_+	100,000	10,000,000,001	10,000,100,000	340	1.15
3.	S_-	100,001	10,000,100,001	10,000,200,001	-249	0.620
4.	S_+	100,001	10,000,400,005	10,000,500,006	-115	0.132
5.	S_-	100,002	10,000,300,003	10,000,400,004	216	0.467
6.	S_+	100,002	10,000,400,005	10,000,500,006	456	2.08
7.	S_-	100,003	10,000,500,007	10,000,600,009	-255	0.650
8.	S_+	100,003	10,000,600,010	10,000,700,012	-235	0.552
9.	S_-	100,004	10,000,700,013	10,000,800,016	-44	0.0194
10.	S_+	100,004	10,000,800,017	10,000,900,020	202	0.408
11.	S_-	100,005	10,000,900,021	10,001,000,025	-191	0.364
12.	S_+	100,005	10,001,000,026	10,001,100,030	475	2.26
13.	S_-	100,006	10,001,100,031	10,001,200,036	134	0.179
14.	S_+	100,006	10,001,200,037	10,001,300,042	-66	0.0436
15.	S_-	100,007	10,001,300,043	10,001,400,049	427	1.82
16.	S_+	100,007	10,001,400,050	10,001,500,056	-303	0.918
17.	S_-	100,008	10,001,500,057	10,001,600,064	276	0.762
18.	S_+	100,008	10,001,600,065	10,001,700,072	-210	0.441
19.	S_-	100,009	10,001,700,073	10,001,800,081	267	0.713
20.	S_+	100,009	10,001,800,082	10,001,900,090	291	0.847
	MEAN	χ^2 **FROM**	**9,999,900,001 TO**	**10,001,900,090**	=	**0.768**

3.1 Sequences of Fixed Length Arbitrarily Positioned

In this section we consider various segments of consecutive integers of a fixed length but starting from an arbitrary integer. Even here we see that the λs within each segment behave like coin tosses and have the same statistical properties.

We now calculate the χ^2 values of λ-sequences for a sequence S_A of consecutive integers, starting from an arbitrary number N_0 but all of a fixed length M:

$$S_A(N) = \{N_0,\, N_0 + 1,\, N + 2,\, N + 3, ...,\, N_0 + M - 1\} \tag{16}$$

and

$$L(S_A) = \lambda(N_0) + \lambda(N_0 + 1) + \lambda(N_0 + 2) + \lambda(N_0 + 3) + ... + \lambda(N_0 + M - 1) \tag{17}$$

The results, which are summarized in Table 2.1, again show that the λ-sequences are statistically like coin tosses.

TABLE 2.1 Sequence of Consecutive Integers of Type $S_A(N)$ and of Length M = 1000

No	*Type*	M	From N_0	to $N_0 + M - 1$	$L(S_A)$	χ^2
1.	S_A	1000	10,000,001	10,001,000	36	1.296
2.	S_A	1000	12,000,001	12,001,000	28	0.784
3.	S_A	1000	13.000,001	13,001,000	-14	0.196
4.	S_A	1000	15,000,001	15,001,000	10	0.10
5.	S_A	1000	45,000,001	45,001,000	-18	0.324
6.	S_A	1000	47,000,001	47,001,000	-36	1.296
7.	S_A	1000	56,000,001	56,001,000	24	0.576
8.	S_A	1000	70,000,001	70,001,000	-44	1.936
9.	S_A	1000	90,000,001	90,001,000	14	0.196
10.	S_A	1000	95,600,001	95,601,000	28	0.784
11.	S_A	1000	147,000,001	147,001,000	-26	0.676
12.	S_A	1000	237,000,001	237,001,000	-24	0.576
13.	S_A	1000	400,000,001	400,001,000	26	0.676
14.	S_A	1000	413,000,001	413,001,000	10	0.10
15.	S_A	1000	517,000,001	517,001,000	14	0.196
16.	S_A	1000	530,000,001	530,001,000	-32	1.024
17.	S_A	1000	731.000,001	731,001,000	50	2.500
18.	S_A	1000	871,000,001	871,001,000	-42	1.764
19.	S_A	1000	979,000,001	979,001,000	-20	0.400
20.	S_A	1000	997,000,001	997,001,000	14	0.196
		MEAN	**χ^2 OF ABOVE**	**20 SEGMENTS**	=	**0.780**

3.2 Entire Sequences from $n = 1$ to $n = N$, N large and calculation of χ^2 for such sequences from $L(N)$

It has been empirically verified in the literature that the summatory Liouville Function $L(N) = \sum_{n=1}^{N} \lambda(n)$ fluctuates from positive to negative values as N increases without bound. We now investigate the χ^2 values for such sequences,and use Eq.(9), so that we may see how these sequences behave like coin tosses.

In the Table 3.1 we use the values of $L(N)$ for various large values of N, which were found by Tanaka (1980), the results depicted below reveal that the lambda sequences are statistically indistinguishable from the sequences of coin tosses over such large ranges of N from 1 to one billion.

In the above we calculated $L(N)$ for various valies of N, however, if we choose a value N at which $L(N)$ is a local maximum or a local minimum then we would be examining potential worst case scenarios for deviations of the λs from coin tosses because these are the values of N that are likely to yield the highest values of χ^2 (see equation (9)). It is interesting to investigate if even for these special values of N whether the χ^2 is less than the critical value; if so, we would again have statistical assurance that the entire sequence of λs from $n = 1, 2, 3, ...$ behave like coin tosses.

We therefore use the 58 largest values of $L(N)$ and the associated values of N reported in the literature by Borwein, Ferguson and Mossinghoff (2008), and perform our statistical exercise. See Table 3.2. We see that even for these "worst case scenario" values of N the lambda sequences are statistically indistinguishable from the sequences of coin tosses.

TABLE 3.1 Values of L(N) at various large values of N
(The values for N and L(N) are from Tanaka (1980))

No.	N	$L(N) = \sum_{n=1}^{N} \lambda(n)$	χ^2
1	100,000,000	-3884	0.1508
2	200,000,000	-11126	0.6189
3	300,000,000	-16648	0.9238
4	400,000,000	-11200	0.3136
5	500,000,000	-18804	0.7072
6	600,000,000	-15350	0.3927
7	700,000,000	-25384	0.9204
8	800,000,000	-19292	0.4652
9	900,000,000	-4630	0.0238
10	1,000,000,000	-25216	0.6358
	MEAN	χ^2 **OF ABOVE =**	0.5152

TABLE 3.2 Values of L(N) at local Minima (Maxima) for very Large N
(The values for N and L(N) are from Borwein, Ferguson and Mossinghoff (2008))

No.	N	$L(N) = \sum_{n=1}^{N} \lambda(n)$	χ^2
1	293	-21	1.5051
2	468	-24	1.2308
3	684	-28	1.1462
4	1,132	-42	1.5583
5	1,760	-48	1.3091
6	2,804	-66	1.5535
7	4,528	-74	1.2094
8	7,027	-103	1.5097
9	9,840	-128	1.665
10	24,426	-186	1.4164
11	59,577	-307	1.582
12	96,862	-414	1.7695
13	386,434	-698	1.2608
14	614,155	-991	1.5991
15	925,985	-1,253	1.6955
16	2,110,931	-1,803	1.54
17	3,456,120	-2,254	1.47
18	5,306,119	-2,931	1.619
19	5,384,780	-2,932	1.5965
20	8,803,471	-3,461	1.3607

TABLE 3.2 (Cont'd) Values of L(N) at local Minima (Maxima)

No.	N	$L(N) = \sum_{n=1}^{N} \lambda(n)$	χ^2	
21	12,897,104	-4,878	1.845	
22	76,015,169	-10,443	1.4347	
23	184,699,341	-17,847	1.7245	
24	281,876,941	-19,647	1.3694	
25	456,877,629	-28,531	1.7817	
26	712,638,284	-29,736	1.2408	
27	1,122,289,008	-43,080	1.6537	
28	1,806,141,032	-50,356	1.4039	
29	2,719,280,841	-62,567	1.4396	
30	3,847,002,655	-68,681	1.2262	
31	4,430,947,670	-73436	1.2171	
32	6,321,603,934	-96,460	1.4719	
33	10,097,286,319	-123,643	1.514	
34	15,511,912,966	-158,636	1.6223	
35	24,395,556,935	-172,987	1.2266	
36	39,769,975,545	-238,673	1.4324	
37	98,220,859,787	-365,305	1.3586	
38	149,093,624,694	-461,684	1.4296	
39	217,295,584,371	-598,109	1.6463	
40	341,058,604,701	-726,209	1.5463	
41	576,863,787,872	-900,668	1.4062	
42	835,018,639,060	-1,038,386	1.2913	
43	1,342,121,202,207	-1,369,777	1.398	
44	2,057,920,042,277	-1,767,635	1.5183	
45	2,147,203,463,859	-1,784,793	1.4836	
46	3,271,541,048,420	-2,206,930	1.4888	
47	4,686,763,744,950	-2,259,182	1.089	
48	5,191,024,637,118	-2,775,466	1.4839	
49	7,934,523,825,335	-3,003,875	1.1372	
50	8,196,557,476,890	-3,458,310	1.4591	
51	12,078,577,080,679	-4,122,117	1.4068	
52	18,790,887,277,234	-4,752,656	1.2021	
53	20,999,693,845,505	-5,400,411	1.3888	
54	29,254,665,607,331	-6,870,529	1.6136	
55	48,136,689,451,475	-7,816,269	1.2692	
56	72,204,113,780,255	-11,805,117	1.9301	
57	117,374,745,179,544	-14,496,306	1.7904	
58	176,064,978,093,269	-17,555,181	1.7504	

The empirical evidence provided here is very comprehensive: it examines the statistical behavior of the Liouville function for large segments of consecutive integers (e.g.Table 1.4). We have also considered the entire series of $\lambda(n)$ from the values of $n = 1$ to $n = N = 176$ trillion - as high as any available studies in the literature have gone. And yet, the $\lambda-$sequences consistently show themselves, in rigorous statistical tests, to be indistinguishable from sequences of coin tosses, hence providing overwhelming statistical evidence in support of Littlewood's condition that as $N \to \infty$, $L(N) = C.\sqrt(N)$, (where C is finite) and thus declaring that the non-trivial zeros of the zeta function, $\zeta(s)$, must all necessarily lie on the critical line $Re(s) = 1/2$.

4. Concluding Note

In this Appendix VI we have provided compelling, comprehensive numerical and statistical evidence that is consistent with the Theorems that were instrumental in the formal validation of the Riemann Hypothesis in [MP and A's].

It is hoped that a perusal of this section (Appendix 6)report offers some insight into, and understanding of, why the Riemann Hypothesis is correct. It should be noted that, while the results presented here are perfectly consistent with the theoretical results in [MP and A's], they obviously do not prove (in a strict mathematical sense, because of the statistical nature of the study), the Riemann Hypothesis. For the formal proof, the rigorous mathematical analysis in the main paper needs to be consulted.

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The Pathway to the Riemann Hypothesis

Preprint · March 2019

DOI: 10.13140/RG.2.2.17950.59208

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The Pathway to the Riemann Hypothesis

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March 20, 2019

This brief Note describes the scheme and the essential steps in the proof of the RH as given by K.Eswaran in his Main Paper.:

"The Final and Exhaustive Proof of The Riemann Hypothesis from First Principles". See Ref [1] (in Reference List below).

This write up also provides an answer to queries raised by readers and other researchers.

The proof involves five basic steps which are descibed below:

1 Step 1:

To look for an analytic function, F(s), whose poles exactly correspond to the non-trivial zeros of the zeta function $\zeta(s)$, so that various mathematical techniques discovered in the field of complex function theory can be used to analize F(s).

It was then found that the function F(s) which is defined as below:

$$F(s) = \frac{\zeta(2s)}{\zeta(s)} \tag{1}$$

has all the above required properties: F(s) has poles at exactly the same positions as $\zeta(s)$ has its non-trivial zeros in the critical region, (the trivial zeros, $s = -2n$ (n: an integer), cancel out and do not appear as poles in F(s); also for F(s), there is one additional pole which corresponds to the single pole in $\zeta(2s)$, but this does not disturb or play any role in our analysis, because it occurs any way on the critical line $s = 1/2$).

F(s) is analytic in the region $Re(s) > 1$and is given by:

$$F(s) = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} \tag{2}$$

2 Step 2:

In this step the necessary and sufficient conditions for the analyticity of F(s) is determined. When a technique used by Littlewood for determining the behaviour of the function $1/\zeta(s)$ by analytic continuation is used to find the behaviour of F(s). It will then be demonstrated that the summatory function L(N) which is defined by:

$$L(N) = \sum_{n=1}^{N} \lambda(n) \tag{3}$$

plays a crucial role in determining the position of the poles of F(s) and thereby the zeros of $\zeta(s)$ in the critical region 0< Re(s) <1. Littlewood's theorem states that the asymptotic behaviour of $L(N)$ for large N, determines the analyticity of F(s), and if the behaviour is such that

$$\mid L(N) \mid \equiv \mid \sum_{n=1}^{N} \lambda(n) \ \mid < C\, N^{a+\epsilon} \qquad (for\ large\ N\,) \tag{4}$$

(where $(1/2 \leq a < 1\,)$, and ϵ is a small positive number), F(s) will be analytic in the region $(a < Re(s)$. This is a very crucial result as far as RH is concerned because if one can determine that actually $a = 1/2$ in (4) then the Riemann Hypothesis is proved.

3 Step 3:

In this step we logically put forth the argument: *that the very necessity that (4) must be satisfied for the Riemann Hypothesis to be true, imposes very severe resrtictions on the behaviour of the sequence of the Liouville functions:* $\{\lambda(1), \lambda(2), \lambda(3), ...\}$. These restrictions (conditions) will be delineated later in this section.

The $\lambda(n)$ is defined as: $\lambda(1) = 1$and for $n > 1$: $\lambda(n) = (-1)^{\Omega(n)}$ and is determined by factorizing n and finding $\Omega(n)$, the number of prime factors of n (multiplicities included). We already know $\lambda(n)$ is fully determined by factorizing n and is an arithmetic function namely: $\lambda(m.n) = \lambda(m).\lambda(n)$, for all integers m, n.

Now for RH to be true one must have $a = 1/2$ in Eq.(4), the first N terms (N large) of the λ sequence must therefore sum up as:

$$\mid \lambda(1) + \lambda(2) + \lambda(3) + ... + \lambda(N) \mid \simeq \ C\,.\,N^{1/2} \tag{5}$$

The above equation brings to mind a similar relationship satisfied by another sequence $c(n) = \pm 1$, which corresponds to the nth step of a One-dimensional random walk! (This $c(n)$ can be simulated by coin tosses, if we replace Heads by +1 and Tails by -1; so a N-step random walk can be thought as a coin toss experiment where a coin is tossed N times.) It is well known that for such a random-walker's sequence the sum indicates the distance travelled from the starting position in N steps and satisfies the relationship:

$$\mid c(1) + c(2) + c(3) + ... + c(N) \mid \simeq \ C\,.\,N^{1/2} \tag{6}$$

Equation (6) is derived by using the assumption that the random walker behaves in such a manner that:

(i) Each step is of the same size but can be either in the positive direction or negative direction i.e the nth step $c(n)$ can be $+1$ or -1,

with Equal Probability.

(ii) The sequence of steps cannot be periodic, that is the pattern of steps cannot form a repetitive pattern.

(iii) Knowing the n^{th} step the $(n+1)^{th}$ cannot be predicted. That is, knowing c(n), c(n+1) cannot be determined (they are independent).

The above assumptions are enough to derive Eq(6). This has been shown by many researchers (e.g. See S. Chandrasekar, refered in Ref[1])

3.1 The Argument:

Comparing (5) and (6) leads us to deduce some inevitable conclusions:

Eq.(5) must be satisfied by the $\lambda(n)$ sequence *if the Riemann Hypothesis is TRUE*, this is the conclusion that we deduce from Littlewoods Theorem. However, (5) needs be satisfied only for large N (this being the condition of Littlewood's theorem). Now, there are many Random walks possible, for instance: 100 random walkers can each of them, take N steps and each of these random walkers will be at anapproximate distance of distance $C \,.\, N^{1/2}$from the starting point. Each of these 100 sequences can be thought of as *100 different instances of a random walk of N steps each.*

If we wish to compare (6) with (5) there are several conceptual issues: (i) The sequence in (5) is a deterministic sequence and (ii) we have only one sequence. We get over this latter issue by considering the single sequence as *one instance* of a hypothetical random walk of N steps. And even though the $\lambda(n)$'s are deterministic (an aspect we temporarily ignore) we could investigate *this one instance* and argue (or hypothesize) that when N is large, the following rules could be obeyed:

Properties of the $\lambda-$sequence

(a) Given an arbitrary large n chosen at random, there is Equal probability of $\lambda(n)$being either $+1$ or-1.

(b) The $\lambda-$sequence cannot be periodic, that is the $\lambda(n)$ cannot form a repetitive pattern (no cycle)

(c) Knowing the value of $\lambda(n)$ it is not possible to predict $\lambda(n+1)$. Unpredictability (independence).

Note the rules (a),(b)and (c) are similar to (i),(ii) and (iii) and therefore: ***If by using the number theoretical (arithmetical) properties of the integers, the primes and the factorization process, it is somehow possible to prove that the $\lambda(n)$ satisy the rules (a), (b) and (c) then just as (6) is satisfied by every instance of a random walk, Eq(5) will be satisfied for our one particular instance of our $\lambda-$sequence and thus RH will be proved! Taking this as a cue we proceed.***

Hence the next step is to prove the properties for large values of N i.e. when N tends to infinity. (It will become clear later that the deterministic nature of the $\lambda(n)'s$, does not significantly disturb the above statistical properties.[1].)

4 Step 4: Proof of the Properties of the $\lambda-$sequence

In this step several theorems are proved using the number theoretical (arithmetical) properties of integers, primes and the unique factorization of integers to establish the properties (a), (b) and (c) of the $\lambda-$sequence as listed in the previous paragraphs. These proofs are fairly straight forward and are from first principles:

Property (a) On Equal Probabilities, is proved in Theorems 2 and 3 in Section 5.2, in the Main Paper Ref [1]. The concept of "towers" is used in the proofs. An alternative proof[2] by constuction of all prime products and induction is also given in Ref [2]. A third proof, which follows from Littlewood's theorem but assumes the fact that there is no zero with Re(s)=1, (proved in the Prime Number Theorem) can also be derived (but is not given in the paper).

Property (b) On no cycles, is proved in Appendix III, Ref [1]

Property(c) On unpredictability (independence) is proved in Appendix IV, Ref[1]. An alternative proof of this also given: See para 5(a), in page 2, of Ref [3].

An alternative arithmetical proof of the asymptotic relation $\mid L(N) \mid \simeq c = C.N^{1/2}$. is given in Appendix V, Ref [1].[3]

The above extablishes RH.

Finally, we show that the 'width' of the Critical Line' must necessarily tend to zero. See: 3rd paragraph before the Conclusion in Ref[1].

5 Experimental verification

In the last Appendix VI, Ref[1], numerical experiments (using Mathematica) are described and there it is shown that large sequence of lambdas behave like

[1]For example, given $\lambda(n)$ for some n, the formula $\lambda(m.n) = \lambda(m).\lambda(n)$,can determine the next predictable value $\lambda(2.n) = \lambda(2).\lambda(n) = -\lambda(n)$, but for large n, say $n = 10^{100}$, the integer $2n$ will be at a distance of 10^{100} from n making such a prediction statistically insignificant!

[2]This alternative proof of equal probabilities is given in Ref [2] and is done by explict construction of integers by products of sets which are powers of a given prime. This is from first principles using the principle of mathematical induction and all that is assumed: is that every odd number is succeded by an unique even number which is preceded by an unique odd number.

[3]In a separate arithmetical study Ref [4], it was discovered that for very large N, smaller primes contribute more (than the larger primes) to the calculation of $\lambda(n)$'s which occur in the summatory expression for $L(N)$. Specifically, if one chooses an integer K such that $K << N$ then the primes p which are s.t $p < N/K$, occur much more often in the calculation of each term in $L(N)$ than the large primes q which are s.t. $N/K < q < N$. This situation permits us to deduce, interestingly, that if we allow both $K\, and\, N \to \infty$ in such a manner that the ratio N/K is a fixed number, then we must have: $Pr(\lambda(n) = 1 \mid n < N) = 1/2 - \frac{C_K}{logN}$ and $Pr(\lambda(n) = -1 \mid n < N) = 1/2 + \frac{C_K}{logN}$ where $C_K$is a small fluctuating number which tends to zero as $K \to \infty$; thus once again confirming that the $L(N)$ behaves like a random walk for very large N.

a random walk (or equivalently like coin tosses). These sequences called $S_-(N)$ and $S_+(N)$ exist (N being a perfect square) and behave like random sequences (coin tosses) and the concatination of such sets of $S_-(N)$ and $S_+(N)$ cover all of $\lambda(n)$ for all integers n upto infinity. The Tables given in this Appendix VI, provide ample proof of this, for instance, see Table 1.4. The purpose of this section is just to demonstrate that the, predictions of the theorems proved have been numerically verified extensively. The verification has been done by doing a χ^2 fit of a $\lambda-$sequence with a Binomial distribution (coin tosses). In every case it has been shown that for large N the $\lambda-$sequence is indistinguishable from a random walk (sequence of coin tosses).

6 References

[1] The final and Exhaustive proof of the Remann Hypothesis...

[2]A Simple Proof That Even and Odd Numbers of Prime Factors Occur with Equal Probabilities in the Factor-ization of Integers

[3]A Quick Reading Guide to the Proof of the Riemann Hypothesis

[4] The effect of of the non-random-walk behavior of the Liouville Series L(N) by the first finite number of terms.

I enclose below the slides of the Invited Lecture that I delivered at the Government Arts & Science College Kumbakonam on March 1st 2019. (This was followed by another (slightly shorter) Lecture delivered in the Ramnujan Centre of Sastra University on the evening of the same day).

[5] Invited Lecture On the Riemann Hypothesis by K.Eswaran

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20/3/2019

A Simple Proof That Even and Odd Numbers of Prime Factors Occur with Equal Probabilities in the Factor-ization of Integers and its Implications for the Riemann Hypothesis

Preprint · April 2018

DOI: 10.13140/RG.2.2.28975.43684/2

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A Simple Proof That Even and Odd Numbers of Prime Factors Occur with Equal Probabilities in the Factorization of Integers and its Implications for the Riemann Hypothesis

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ABSTRACT: In this paper it is demonstrated that that the probability that an integer has an even number of prime factors (multiplicity included) is equal to the probability that it has an odd number of prime factors. This result, which was already proved alternatively in [6], is closely connected to the Riemann Hypothesis (RH) and provides a route to the proof of RH. The method followed in [6] to proving RH is briefly outlined.

1 The Result

In the following, we construct the integers from the prime numbers by reverse engineering the Prime Factorization Theorem. This provides a powerful and remarkably simple method for demonstrating that the probability that an integer has an even number of prime factors (multiplicity included) is equal to the probability that it has an odd number of prime factors. This finding is closely connected with the Riemann Hypothesis (RH) and is necessary for its proof.

We use the Liouville function [2], $\lambda(n)$, defined by $\lambda(n) = (-1)^{\Omega(n)}$, where $\Omega(n)$ is the total number of prime numbers in the factorization of n, including the multiplicity of the primes. By definition, $\lambda(1) = 1$. The result we prove states that if n is an arbitrary integer, then

$$Prob[\lambda(n) = +1] = Prob[\lambda(n) = -1] = 1/2.$$

To demonstrate this result, we recursively construct infinite sets of positive integers labeled by I_j, where j runs through the prime numbers: $2, 3, 5, 7, ...$

First consider the set I_2 defined as

$$I_2 = \{2^k, \quad k = 0, 1, 2, 3, ...\}.$$

The integers in each of the pairs in this set $(1, 2), (2^2, 2^3), (2^4, 2^5), ...$ are 'twins' in the sense that they have λs of opposite signs:

$$\lambda(2^{2k}) = -\lambda(2^{2k+1}), \quad k = 0, 1, 2, 3, ...$$

An ordered countably infinite set will be equipartitioned if its members in order are alternately placed in two different sub-sets. So we conclude that there are equal proportions of integers with λ's of +1 and -1 in I_2.

Next, we introduce the prime 3 by defining the set I_3:

$$I_3 = \{3^l a_k, \quad a_k \in I_2 \ and \ l = 0, 1, 2, 3, ...\}.$$

This set comprises members of the form
$\{1, 2, 2^2, 2^3, ...$
$3, 3 \times 2, 3 \times 2^2, 3 \times 2^3, 3 \times 2^3, ...$
$3^2, 3^2 \times 2, 3^2 \times 2^2, 3^2 \times 2^3, 3^2 \times 2^3, ...$
$3^3, 3^3 \times 2, 3^3 \times 2^2, 3^3 \times 2^3, 3^3 \times 2^3, ...$
$3^4, 3^4 \times 2, 3^4 \times 2^2, 3^4 \times 2^3, 3^4 \times 2^3, ...\}$

In this set, all the pairs on the first line are twins, as we have seen. In each of the remaining lines, $(3^{2l-1}a_k, 3^{2l}a_k), l = 1, 2, 3, ...$ are twins because they have λs of opposite signs. That is any row which contains a particular number which contains the factor 3^{2l-1} will have as its 'twin' the number just below it which contains the factor 3^{2l}. So we conclude that there are equal proportions of integers with λ's of +1 and -1 in I_3.

In a similar manner, we can introduce the primes $5, 7, ...$etc. Suppose p is an arbitrary prime and I_p is the corresponding set recursively constructed as shown above. If q is the next prime after p, we define

$$I_q = \{q^l a_k, \quad \forall a_k \in I_p \ and \ l = 0, 1, 2, 3, ...\}. \tag{1}$$

Note that I_q contains all the integers that can be constructed using primes $\leq q$. Therefore, I_q contains all the integers belonging to I_m for $m < q$.

Theorem A: All the integers in the set I_q, $q = 2, 3, 5, 7, ...$can be paired so that their λ-values are of the opposite sign.

Proof: Since the claim has already been demonstrated above for $q = 2$, we presume that $q > 2$. Let p be the prime that immediately precedes q, and suppose that all the integers in the set I_p are twinned.

When $l = 0$ in Eq. (1) all the integers in I_q are a_k, which belong to I_p are twinned by assumption. When $l > 0$, for every $a_k \in I_p$, the pairs $(q^{2l-1}a_k, q^{2l}a_k)$, $l = 1, 2, 3, ...$ are obviously twins in the sense that

$$\lambda(q^{2l-1}a_k) = -\lambda(q^{2l}a_k), \quad l = 1, 2, 3, ...$$

Therefore, if the integers in I_p are twinned, so are the integers in I_q. But we have seen that when $p = 2$, all the integers in I_2 are twinned. Thus the property claimed in this Theorem follows by induction. QED. [1]

Continuing in this manner, we see that all the integers in I_∞ can be twinned. But I_∞ comprises the entire set of positive integers. Therefore we have shown

[1] There are of course, other methods of 'twinning' : If we choose any two numbers $a, b \ \varepsilon \ I_p$ such that (a, b) are twins in I_p then the two numbers $(q^m.a, q^m.b)$ are twins in I_q for any $m \geq 1$ (Remember p is a prime and q is the next higher prime).

that the natural numbers have equal proportions with λ's of +1 and -1. Thus we have shown:

Theorem B: If n is an arbitrary positive integer,

$$Prob[\lambda(n) = +1] = Prob[\lambda(n) = -1] = 1/2. \tag{2}$$

The Connection to the Riemann Hypothesis

The condition of equal probabilities proved above is only a necessary condition. The necessary and sufficient condition for RH to be valid is that for large N: [3],[4]-[7]:

$$\frac{L(N)}{N^{\frac{1}{2}+\epsilon}} = 0 \ as \ N \to \infty, \tag{3}$$

for $\epsilon > 0$, where $L(N)$ is the so called Liouville summatory function defined by

$$L(N) = \sum_{n=1}^{N} \lambda(n). \tag{4}$$

The Steps Used to Prove the Riemann Hypothesis

The steps undertaken to prove the Riemann hypothesis are detailed in [5] (see also its accompanying guide [6]). We briefly summarize the method that was followed in [5] to establish RH.

1. We demonstrate that the $L(N)$ series above is a standard *random walk*, whose distance travelled in N steps can be represented as:

   $$X(N) = \sum_{i=1}^{N} x_i \tag{5}$$

   where the x_i's are independent random numbers with an equal probability of being either +1 or -1, i.e., essentially "coin-tosses". It is a well-known result that the expected value of $X(N)$, for large N, is

   $$lim_{N\to\infty} E(|X(N)|) = C_0 N^{1/2} \tag{6}$$

2. To do this we have to prove that the $\lambda(n)$'s in the $L(N)$ series are essentially "coin-tosses", for large n. That is, we have first to show that (i) their probabilities of being either +1 of -1 are equal, as was proved by Theorem 3 in [5], and has been shown again, more elegantly, by Theorem B above.

3. Further, we show (ii) that the λ's appearing in the natural sequence, $n = 1, 2, 3, ...$, are independent of each other for large n. This is done by two different approaches in Appendices III and IV in [5].

4. We then invoke (towards the end of Section 5 in [5]) Khinchin and Kolmogorov's law of the iterated logarithm [10]-[11] to show that the maximum deviation d_N, from the one-half power-law expectation, in the exponent of $|X(N)|$ for any *individual* random walk tends monotonically to zero as $N \to \infty$. We have proved above that the $L(N)$ series is *one* realization of a random walk. So Khinchin and Kolmogorov's law also holds for $|L(N)|$. Therefore for any chosen $\epsilon > 0$ in (3), the statement for the validity of RH will be satisfied and the Riemann Hypothesis is proved.

2 Acknowledgements

I thank my brothers Professors Mukesh Eswaran (Univ. of British Columbia, Canada) and Vinayak Eswaran (IIT, Hyderabad) for their many discussions and criticisms and for their careful reading of this manuscript. I thank the management of SNIST for their sustained encouragement and support.

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